Description of a program to compute potential magnetic fields in the solar corona

Part I: The spherical harmonic coefficients.

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abusus non tollit usum

Expansion in spherical harmonics

It will be convenient to review some vector analysis using orthogonal curvilinear coordinates. A vector $\mathbf{a}$ can be decomposed into a sum of components along unit vectors $\mathbf{1}_i$ $(i=1,2,3)$:

$$\mathbf{a} = \sum_{i=1}^{3} a_i \mathbf{1}_i$$

where we just require that the unit vectors be orthogonal and that the coordinate system be right-handed, i.e. that

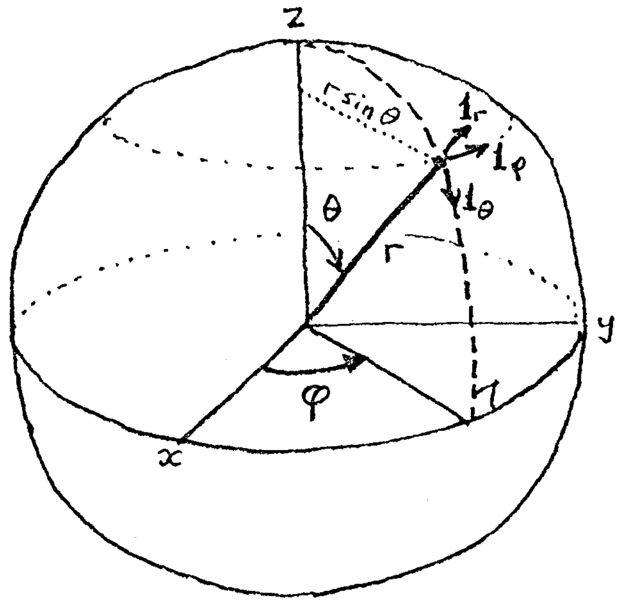
$$\mathbf{1}_k \cdot \mathbf{1}_j = \delta_{kj} \equiv \begin{cases} 1 & ; \ j=k \\ 0 & ; \ j \neq k \end{cases}$$

$$\mathbf{1}_1 \times \mathbf{1}_2 = \mathbf{1}_3$$

The coordinate axes need not be straight lines. The only requirement is that specification of a value for each of three coordinates ξ_1, ξ_2, ξ_3 be sufficient to locate a point in space. An increment $d\xi$ in any coordinate does not in general correspond to a length displacement - often a coordinate will be an angle for example - in general a scale factor h is required such that an increment of length along the ith coordinate is $dl_i = h_i d\xi_i$. The scale factors are in general functions of position, e.g. $h_1 = h_1(\xi_1, \xi_2, \xi_3)$. For the usual Cartesian coordinate system:

$$h_1 = h_2 = h_3 = 1$$

We shall be interested in a spherical coordinate system, where



$r \equiv$ radial distance $(r \geq 0)$

$\theta \equiv$ polar angle $(0 \leq \theta \leq \pi)$

$\varphi \equiv$ azimuth angle $(0 \leq \varphi < 2\pi)$

From the Figure it follows that the scale factors are

$$h_r = h_1 = 1$$
$$h_\theta = h_2 = r$$
$$h_\varphi = h_3 = r\sin\theta$$

The gradient of a scalar point function $F(\zeta_1, \zeta_2, \zeta_3)$ is the vector function that gives the magnitude and direction of the maximum directional derivative of the scalar or

$$\mathbf{grad}\ F = \sum_{i=1}^{3} \frac{\mathbf{1}_i}{h_i} \frac{\partial F}{\partial \zeta_i}$$

The divergence of a vector function $\mathbf{G}(\zeta_1, \zeta_2, \zeta_3)$ is the scalar function defined by

$$\mathrm{div}\ \mathbf{G} = \frac{1}{h_1 h_2 h_3} \sum_{\substack{i \\ k,j \neq i}} \frac{\partial}{\partial \zeta_i}\left(h_k h_j G_i\right)$$

Laplace's equation $\nabla^2 \psi = 0$ can now be written

$\nabla^2 \psi = \mathrm{div}\ \mathbf{grad}\ \psi$

$$= \frac{1}{h_1 h_2 h_3} \sum_{\substack{i \\ j,k \neq i}} \frac{\partial}{\partial \zeta_i}\left(\frac{h_j h_k}{h_i} \frac{\partial \psi}{\partial \zeta_i}\right) = 0$$

For Cartesian coordinates this reduces to

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

Every solution ψ of this equation is called a harmonic function. Many such functions can be written down immediately, viz.

$x, y, z, xy, yz, zx, x^2-y^2, y^2-z^2, z^2-x^2, \ldots$

all satisfying $\nabla^2\psi = 0$. All these functions are of homogeneous degree in x, y, z. We can express $x, y,$ and z in terms of spherical coordinates:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta .$$

A harmonic function of homogeneous degree n in x, y, z can then be expressed as $r^n S_n(\theta, \varphi)$. Such a function is called a spherical harmonic function of degree n; n may be positive or negative but is considered to be an integer. Of course, not every homogeneous expression of degree n is a solution of $\nabla^2\psi = 0$. This expression imposes certain limitations on the harmonic function as we shall see. The function $S_n(\theta, \varphi)$ obviously involves θ and φ through $\sin\theta$ and $\cos\theta$ to the degree n at most, and $\sin\varphi$ and $\cos\varphi$ likewise. Powers of $\sin\varphi$ and $\cos\varphi$ can be transformed into sines and cosines of integral multiples of φ. For example, we have

$$\cos^5\varphi = \frac{1}{16}\left(\cos 5\varphi + 5\cos 3\varphi + 10\cos\varphi\right)$$

$$\sin^5\varphi = \frac{1}{16}\left(\sin 5\varphi - 5\sin 3\varphi + 10\sin\varphi\right)$$

The general expression is quite cumbersome. All that we need here, though, is just the assurance of the fact that we can transform powers of the trigonometric functions into expressions involving multiples of the angle. Collecting terms involving $m\varphi$ we may express S_n in the form

$$S_n = \sum_{m=0}^{n} S_{n,m}(\theta)\cos(m\varphi + \varepsilon_n^m)$$

where the coefficients $S_{n,m}$ are functions of θ only. The phase-angles ε_n^m just serve to eliminate the sine-terms.

Writing Laplace's equation in spherical coordinates we immediately get (with the proper scale factors)

$$\nabla^2\psi = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\varphi}\right)\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}$$

$$= 0$$

By substituting $\psi = r^n S_n(\theta, \varphi)$ in this expression the result becomes

$$r^{n-2}\left\{n(n+1)S_n + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S_n}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 S_n}{\partial\varphi^2}\right\} = 0$$

or by dividing r^{n-2} away and setting $\cos\theta = \mu$:

$$n(n+1)S_n + \frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial S_n}{\partial\mu}\right) + \frac{1}{1-\mu^2}\frac{\partial^2 S_n}{\partial\varphi^2} = 0 \quad , \text{ or}$$

$$(1-\mu^2)\frac{\partial^2 S_n}{\partial\mu^2} - 2\mu\frac{\partial S_n}{\partial\mu} + n(n+1)S_n + \frac{1}{1-\mu^2}\frac{\partial^2 S_n}{\partial\varphi^2} = 0$$

This is the condition a function S_n must satisfy in order that it be a solution to Laplace's equation. The degree n of the harmonic enters through the term $n(n+1)S_n$ only. This term is unchanged if n is changed to $-(n+1)$, so that if $r^n S_n$ is a solution, so also is $r^{-(n+1)}S_n$.

On substituting the expression for S_n in terms of $S_{n,m}$ and cosine of multiples of φ into the differential equation above, we get

$$\sum_{m=0}^{n}\left\{(1-\mu^2)\frac{\partial^2 S_{n,m}}{\partial\mu^2} - 2\mu\frac{\partial S_{n,m}}{\partial\mu} + \right.$$

$$\left. + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]S_{n,m}\right\}\cos(m\varphi + \varepsilon_n^m) = 0$$

Because of the uniqueness of the Fourier expansion the factor of each term of $\cos(m\varphi+\varepsilon_n^m)$ must vanish separately for the whole sum to vanish for all values of φ. Hence we get that the functions $S_{n,m}$ must satisfy:

$$(1-\mu^2)\frac{\partial^2 S_{n,m}}{\partial\mu^2} - 2\mu\frac{\partial S_{n,m}}{\partial\mu} + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]S_{n,m} = 0$$

It is straightforward - although rather tedious - to show that the associated Legendre functions $P_n^m(\mu)$ multiplied by arbitrary constants C_n^m satisfy just this type of condition. We therefore have that the general harmonic function expressible in integral powers of x, y, and z is a sum of harmonics $r^n S_n(\theta, \varphi)$ of all degrees n, namely

$$\sum_{n=0}^{\infty} r^n \sum_{m=0}^{n} C_n^m P_n^m(\mu) \cos(m\varphi + \varepsilon_n^m)$$

As we saw above, if $r^n S_n$ is a solution so is also $r^{-(n+1)} S_n$; we thus establish that the general harmonic function expressible in integral powers of r^{-1} is

$$\sum_{n=0}^{\infty} r^{-(n+1)} \sum_{m=0}^{n} C_n^m P_n^m(\mu) \cos(m\varphi + \varepsilon_n^m)$$

We note that the 'm' in P_n^m, etc, is a superscript and not an exponent.

Consider a sphere with radius a and a potential ψ over the surface. If the potential arises from within the sphere only it must vanish at infinity and therefore no positive powers of r can occur in the expansion for ψ; hence we may express ψ for $r \geqslant a$ as

$$\psi_{in} = a \sum_{n=0}^{\infty} (a/r)^{n+1} S_n(\theta, \varphi)$$

We have inserted the factor a (the radius of the sphere) to give the arbitrary constants in S_n the physical dimension of force. If we consider solely a potential arising from outside the sphere then ψ must be finite throughout the interior and so no negative powers of r can appear in the expansion for ψ. Hence for $r \leq a$:

$$\psi_{out} = a \sum_{n=0}^{\infty} (r/a)^n S_n(\theta, \varphi)$$

Let us now consider two concentric spheres with radii r_0 and r_s ($r_0 \leq r_s$) respectively. The potential between them arises from sources either inside or outside the spherical shell. We write thence:

$$\psi_{in} = r_0 \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(\mu) \left(\frac{r_0}{r}\right)^{n+1} \left\{A_{ni}^m \cos m\varphi + B_{ni}^m \sin m\varphi\right\}$$

$$\psi_{out} = r_s \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(\mu) \left(\frac{r}{r_s}\right)^{n} \left\{A_{no}^m \cos m\varphi + B_{no}^m \sin m\varphi\right\}$$

and hence

$$\psi = \psi_{in} + \psi_{out} = r_0 \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(\mu) \cdot$$

$$\left\{\left[\left(\frac{r_0}{r}\right)^{n+1} A_{ni}^m + \frac{r_s}{r_0}\left(\frac{r}{r_s}\right)^n A_{no}^m\right] \cos m\varphi + \left[\left(\frac{r_0}{r}\right)^{n+1} B_{ni}^m + \frac{r_s}{r_0}\left(\frac{r}{r_s}\right)^n B_{no}^m\right] \sin m\varphi\right\}$$

We can introduce ratios c_n^m and d_n^m so that the above expansion can be written

$$\psi = r_0 \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(\mu) \left\{ \left[\left(\frac{r_0}{r}\right)^{n+1} + \frac{r_s}{r_0}\left(\frac{r}{r_s}\right)^n c_n^m \right] A_n^m \cos m\varphi + \right.$$

$$\left. + \left[\left(\frac{r_0}{r}\right)^{n+1} + \frac{r_s}{r_0}\left(\frac{r}{r_s}\right)^n d_n^m \right] B_n^m \sin m\varphi \right\}$$

Where we have omitted the subscript 'i' on A and B.

For ψ to be zero on the surface of the sphere with $r=r_s$ it is sufficient that

$$\left(\frac{r_0}{r_s}\right)^{n+1} + \frac{r_s}{r_0}\left(\frac{r_s}{r_s}\right)^n c_n^m = \left(\frac{r_0}{r_s}\right)^{n+1} + \frac{r_s}{r_0}(1)^n d_n^m \equiv 0 \text{ , or}$$

$$c_n = d_n = -(r_s/r_0)^{-(n+2)} = -\left(\frac{r_0}{r_s}\right)^{n+2}$$

where the superscript 'm' has been dropped as it does not occur on the righthand side.

As $r_s \to \infty$, the value of c_n tends to zero as expected when we move the external sources to infinity. If $r_s \to r_0$ we see that $c_n \to -1$ just expressing the fact that the internal and external sources must give rise to cancelling potentials. We caution that the c_n^m and d_n^m are not the same as used in Chapman and Bartels (1940) or by previous workers (Altschuler and Newkirk, 1969; Altschuler et al., 1977; Riesebieter, 1977).

To obtain a standard notation we write g_n^m for A_n^m and h_m for B_n^m. Because $c_n = d_n$ we can finally write:

$$\psi = r_0 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[\left(\frac{r_0}{r}\right)^{n+1} + \frac{r_s}{r_0}\left(\frac{r}{r_s}\right)^{n} c_n \right] P_n^m(\mu) \cdot \left\{ g_n^m \cos m\varphi + h_n^m \sin m\varphi \right\}$$

The components of the field (B_1, B_2, B_3) are now given by

$$B_i = -\frac{1}{h_i}\frac{\partial \psi}{\partial \zeta_i}$$

i.e. $B_r = -\dfrac{\partial \psi}{\partial r}$, $B_\theta = -\dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta}$, $B_\varphi = -\dfrac{1}{r\sin\theta}\dfrac{\partial \psi}{\partial \varphi}$.

We obtain now after some minimal manipulation:

$$B_r(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} \left[(n+1)\left(\frac{r_0}{r}\right)^{n+2} - n c_n \left(\frac{r}{r_s}\right)^{n-1} \right] \cdot P_n^m(\mu)\left(g_n^m \cos m\varphi + h_n^m \sin m\varphi\right)$$

$$B_\theta(r,\theta,\varphi) = -\sum_{n=0}^{\infty}\sum_{m=0}^{n} \left[\left(\frac{r_0}{r}\right)^{n+2} + c_n \left(\frac{r}{r_s}\right)^{n-1} \right] \cdot \frac{\partial P_n^m(\mu)}{\partial \theta}\left(g_n^m \cos m\varphi + h_n^m \sin m\varphi\right)$$

$$B_\varphi(r,\theta,\varphi) = \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=0}^{n} \left[\left(\frac{r_0}{r}\right)^{n+2} + c_n \left(\frac{r}{r_s}\right)^{n-1} \right] \cdot m \cdot P_n^m(\mu)\left(g_n^m \sin m\varphi - h_n^m \cos m\varphi\right)$$

If we ignore for the moment the angle $(\leq 7°)$ between the line-of-sight and the solar equator we can write for the line-of-sight component of the photospheric field:

$$B_\ell(r_0,\theta,\varphi) = B_r(r_0,\theta,\varphi)\sin\theta + B_\theta(r_0,\theta,\varphi)\cos\theta$$

With the above expansions, we obtain:

$$B_\ell = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{u_n \sin\theta\, P_n^m - v_n \cos\theta \frac{\partial P_n^m}{\partial\theta}\right\} F(n,m,\varphi)$$

where $\quad F(n,m,\varphi) = g_n^m \cos m\varphi + h_n^m \sin m\varphi$

$$u_n = (n+1) - n c_n \left(\frac{r_0}{r_S}\right)^{n-1} = (n+1) + n\left(\frac{r_0}{r_S}\right)^{2n+1}$$

$$v_n = 1 + c_n\left(\frac{r_0}{r_S}\right)^{n-1} = 1 - \left(\frac{r_0}{r_S}\right)^{2n+1}$$

By multiplying B_ℓ by $\sin\theta$:

$$B_\ell \sin\theta = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{u_n P_n^m - u_n \cos\theta \cos\theta P_n^m - v_n \cos\theta \sin\theta \frac{\partial P_n^m}{\partial\theta}\right\} F(n,m,\varphi)$$

we can utilize the recurrence relations (e.g. Chapman and Bartels, 1940, p.622):

$$\cos\theta\, P_n^m = \frac{1}{2n+1}\left[\left\{(n+1)^2 - m^2\right\}^{1/2} P_{n+1}^m + \left\{n^2 - m^2\right\}^{1/2} P_{n-1}^m\right]$$

$$\sin\theta \frac{\partial P_n^m}{\partial\theta} = \frac{1}{2n+1}\left[n\left\{(n+1)^2 - m^2\right\}^{1/2} P_{n+1}^m - (n+1)\left\{n^2 - m^2\right\}^{1/2} P_{n-1}^m\right]$$

Note that we have omitted the argument of $P_n^m(\mu)$ for convenience. We can now eliminate the derivative of P_n^m and write for $B_\ell \sin\theta$ at $r = r_0$:

$$B_\ell \sin\theta = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{u_n P_n^m - \frac{1}{2n+1}\left[\left\{(n+1)^2 - m^2\right\}^{1/2}(u_n + n v_n)\right]\cos\theta\, P_{n+1}^m \right.$$

$$\left. - \frac{1}{2n+1}\left[\left\{n^2 - m^2\right\}^{1/2}\left(u_n - (n+1) v_n\right)\right]\cos\theta\, P_{n-1}^m\right\} F(n,m,\varphi)$$

We shall comment below on the occurrence of terms containing P_{n-1}^m for $n = 0$.

Once again we can apply the recurrence relations to eliminate the $\cos\theta$ factors. Further simplification results from the fact that

$$u_n + n v_n = 2n+1$$

$$u_n - (n+1) v_n = (2n+1)\left(\frac{r_0}{r_s}\right)^{2n+1}$$

and by introducing the function

$$Q_n^m = \{n^2 - m^2\}^{1/2} \quad , \quad Q_n^n \equiv 0$$

We then arrive at

$$B_\ell \sin\theta = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left\{u_n P_n^m - \frac{Q_{n+1}^m}{2n+3}\left[Q_{n+2}^m P_{n+2}^m + Q_{n+1}^m P_n^m\right] - \right.$$

$$\left. - \frac{Q_n^m}{2n-1}\left(\frac{r_0}{r_s}\right)^{2n+1}\left[Q_n^m P_n^m + Q_{n-1}^m P_{n-2}^m\right]\right\}(g_n^m \cos m\varphi + h_n^m \sin m\varphi)$$

It should be noted that $Q_k^k = 0$ and that $Q_{n-1}^m = 0$ for $n=1, m=0$ thus saving us from defining P_{n-2}^m for $n<2$. We just omit these terms in actual computations while retaining them in the formal expression.

Let us keep m <u>fixed</u> and write out the contribution to the sums for all n:

$$B^m = \sum_{n=m}^{\infty}\left\{\left[u_n - \frac{(Q_{n+1}^m)^2}{2n+3} - \frac{(Q_n^m)^2}{2n-1}\left(\frac{r_0}{r_s}\right)^{2n+1}\right]P_n^m\right.$$

$$\left. - \frac{Q_{n+1}^m Q_{n+2}^m}{2n+3} P_{n+2}^m - \frac{Q_{n-1}^m Q_n^m}{2n-1}\left(\frac{r_0}{r_s}\right)^{2n+1} P_{n-2}^m\right\} \cdot$$

$$\cdot (g_n^m \cos m\varphi + h_n^m \sin m\varphi)$$

We have in effect written

$$B_\ell \sin\theta = \sum_{m=0}^{\infty} B^m$$

We now decompose the sum in B^m into three sums as follows

$$B^m = \sum_{n=m}^{\infty} \alpha_n^m P_n^m (g_n^m \cos m\varphi + h_n^m \sin m\varphi)$$

$$+ \sum_{n=m+2}^{\infty} \beta_n^m P_n^m (g_{n-2}^m \cos m\varphi + h_{n-2}^m \sin m\varphi)$$

$$+ \sum_{n=m-2}^{\infty} \gamma_n^m P_n^m (g_{n+2}^m \cos m\varphi + h_{n+2}^m \sin m\varphi)$$

where

$$\alpha_n^m = u_n - \frac{(Q_{n+1}^m)^2}{2n+3} - \frac{(Q_n^m)^2}{2n-1} \left(\frac{r_0}{r_s}\right)^{2n+1}$$

$$\beta_n^m = - \frac{Q_{n-1}^m Q_n^m}{2n-1} \quad ; \quad \beta_m^m \equiv \beta_{m+1}^m \equiv 0$$

$$\gamma_n^m = - \frac{Q_{n+1}^m Q_{n+2}^m}{2n+3} \left(\frac{r_0}{r_s}\right)^{2n+5}$$

Note that the dummy index n has three <u>different</u> lower bounds - one for each sum. This was done in order to have only P_n^m in the expression. In the last sum the lower index $n = m-2$ could be replaced by $n = m$ because $\gamma_{m-2}^m = \gamma_{m-1}^m \equiv 0$. We will do just that from now on.

The function $B_\ell \sin\theta$ can be developed in spherical harmonics quite generally:

$$B_\ell \sin\theta = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m (a_n^m \cos m\varphi + b_n^m \sin m\varphi)$$

where we have to determine coefficients a_n^m and b_n^m.

With the normalization we use – and shall discuss later – the associated Legendre functions satisfy the following orthogonality relation

$$\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} P_n^m \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\varphi \cdot P_{n'}^{m'} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m'\varphi \cdot d\mu\, d\varphi = \frac{\delta_{nn'}\,\delta_{mm'}}{2n+1}$$

We now evaluate $P_{n'}^{m'}(\theta)\, B_\ell \sin\theta \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m'\varphi$ and integrate over the sphere. Then the result is only non-zero for $n=n'$ and $m=m'$ leading directly to a determination of a_n^m and b_n^m. Using a double sum rather than integrals we obtain simply:

$$\begin{Bmatrix} a_n^m \\ b_n^m \end{Bmatrix} = \frac{2n+1}{NM}\sum_{i=1}^{N}\sum_{j=1}^{M} B_\ell(\theta_i,\varphi_j)\sin\theta_i\, P_n^m(\theta_i) \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m\varphi_j$$

where $B_\ell(\theta_i,\varphi_j)$ is the observed field at the point (θ_i,φ_j). The surface is divided into equal area regions – N in colatitude and M in longitude – each specified through effective colatitude and longitude.

By noting that coefficients to $\cos m\varphi$ and $\sin m\varphi$ must be equal in both expressions for $B_\ell \sin\theta$ we obtain the following relations:

$$a_n^m = \alpha_n^m g_n^m + \beta_n^m g_{n-2}^m + \gamma_n^m g_{n+2}^m$$

$$b_n^m = \alpha_n^m h_n^m + \beta_n^m h_{n-2}^m + \gamma_n^m h_{n+2}^m$$

We now wish to solve for g and h for all n and m up to a certain maximum principal index $n = T$.

Let us first write out the above relations explicitly, say for $m=0$:

$$a_0^0 = \alpha_0^0 g_0^0 + \beta_0^0 g_{-2}^0 + \gamma_0^0 g_2^0 \qquad ; \text{ remember } \beta_0^0 = 0$$

$$a_1^0 = \alpha_1^0 g_1^0 + \beta_1^0 g_{-1}^0 + \gamma_1^0 g_3^0 \qquad ; \text{ remember } \beta_1^0 = 0$$

$$a_2^0 = \alpha_1^0 g_2^0 + \beta_2^0 g_0^0 + \gamma_2^0 g_4^0$$

$$a_3^0 = \alpha_3^0 g_3^0 + \beta_3^0 g_1^0 + \gamma_3^0 g_5^0$$

$$a_4^0 = \alpha_4^0 g_4^0 + \beta_4^0 g_2^0 + \gamma_4^0 g_6^0$$

$$a_5^0 = \alpha_5^0 g_5^0 + \beta_5^0 g_3^0 + \gamma_5^0 g_7^0$$

...

We can rearrange the scheme and write

$$
\begin{aligned}
a_0^0 &= \boxed{\alpha_0^0 g_0^0} + 0 g_1^0 + \boxed{\gamma_0^0 g_2^0} + 0 g_3^0 + 0 g_4^0 + 0 g_5^0 + 0 g_6^0 \\
a_1^0 &= 0 g_0^0 + \boxed{\alpha_1^0 g_1^0} + 0 g_2^0 + \boxed{\gamma_1^0 g_3^0} + 0 g_4^0 + 0 g_5^0 + 0 g_6^0 \\
a_2^0 &= \boxed{\beta_2^0 g_0^0} + 0 g_1^0 + \boxed{\alpha_2^0 g_2^0} + 0 g_3^0 + \boxed{\gamma_2^0 g_4^0} + 0 g_5^0 + 0 g_6^0 \\
a_3^0 &= 0 g_0^0 + \boxed{\beta_3^0 g_1^0} + 0 g_2^0 + \boxed{\alpha_3^0 g_3^0} + 0 g_4^0 + \boxed{\gamma_3^0 g_5^0} + 0 g_6^0 \\
a_4^0 &= 0 g_0^0 + 0 g_1^0 + \boxed{\beta_4^0 g_2^0} + 0 g_3^0 + \boxed{\alpha_4^0 g_4^0} + 0 g_5^0 + \boxed{\gamma_4^0 g_6^0} \\
a_5^0 &= 0 g_0^0 + 0 g_1^0 + 0 g_2^0 + \boxed{\beta_5^0 g_3^0} + 0 g_4^0 + \boxed{\alpha_5^0 g_5^0} + 0 g_6^0 \\
a_6^0 &= 0 g_0^0 + 0 g_1^0 + 0 g_2^0 + 0 g_3^0 + \boxed{\beta_6^0 g_4^0} + 0 g_5^0 + \boxed{\alpha_6^0 g_6^0}
\end{aligned}
$$

...

We imagine that the scheme continues indefinitely to the right. For reasons of space we have arbitrarily truncated the scheme after $n=6$.

Let us place the source surface at infinity, then $(r_0/r_s)=0$ and $\gamma_n^m \equiv 0$. In this case g_0^0 and g_1^0 follow immediately from a_0^0 and a_1^0, and then

g_2^0 follows from g_0^0, g_1^0 and a_2^0, etc.. We have thus a unique solution for g_n^0 – and by extension for the general case. This is not the case, however, for $r_0/r_S > 0$. But, if we may assume – from physical reasoning or just mathematical necessity – that $\lim_{n\to\infty} g_n^m = 0$, then we can truncate the scheme at some sufficiently high n (say at $n=T$) and consider the scheme as a system of linear equations (extending down to $m=T$, too) which may then be solved for the g_n^m:

$$\mathbf{a}^m = K^m \mathbf{g}^m$$

where the vectors $\mathbf{a}^m$ and $\mathbf{g}^m$ are

$$\mathbf{a}^m = (a_m^m, a_{m+1}^m, a_{m+2}^m, a_{m+3}^m, \dots, a_T^m)$$

$$\mathbf{g}^m = (g_m^m, g_{m+1}^m, g_{m+2}^m, g_{m+3}^m, \dots, g_T^m)$$

and the matrix K^m is

$$K^m = \begin{bmatrix} \alpha_m^m & 0 & \gamma_m^m & 0 & 0 & 0 & \cdots & 0 \\ 0 & \alpha_{m+1}^m & 0 & \gamma_{m+1}^m & 0 & 0 & \cdots & 0 \\ \beta_{m+2}^m & 0 & \alpha_{m+2}^m & 0 & \gamma_{m+2}^m & 0 & \cdots & 0 \\ 0 & \beta_{m+3}^m & 0 & \alpha_{m+3}^m & 0 & \gamma_{m+3}^m \cdots & & \cdots \\ 0 & 0 & \beta_{m+4}^m & 0 & \alpha_{m+4}^m & 0 & \cdots & 0 \\ 0 & 0 & 0 & \beta_{m+5}^m & 0 & \alpha_{m+5}^m & \cdots & \gamma_{T-2}^m \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \beta_T^m & 0 & \alpha_T^m \end{bmatrix}$$

If we let L^m denote the inverse of matrix K^m — we may note that K^m is very well conditioned and can easily be inverted — we finally obtain the solutions we are seeking

$$\mathbf{g}^m = L^m \mathbf{a}^m \,, \quad \mathbf{h}^m = L^m \mathbf{b}^m \,, \qquad m = 0, 1, 2, \ldots, T$$

We may also note that the matrices L^m are of moderate size. For $T=10$, L^0 is an 11×11 matrix, and L^4 is a 7×7 matrix, etc.

Although we have already generalized to the case $m>0$, it is instructive to finish this section by writing out the scheme for $m=4$ and still truncate after $n=6$:

$$a_4^4 = \alpha_4^4 g_4^4 + \beta_4^4 g_2^4 + \gamma_4^4 g_6^4 \quad ; \text{ again: } \quad \beta_4^4 = 0$$
$$a_5^4 = \alpha_5^4 g_5^4 + \beta_5^4 g_3^4 + \gamma_5^4 g_7^4 \quad ; \text{ also: } \quad \beta_5^4 = 0$$
$$a_6^4 = \alpha_6^4 g_6^4 + \beta_6^4 g_4^4 + \gamma_6^4 g_8^4 \quad ; \text{ but: } \quad \beta_6^4 \neq 0$$

We write as before:

$$a_4^4 = \alpha_4^4 g_4^4 + 0 g_5^4 + \gamma_4^4 g_6^4 + 0 g_7^4 + 0 g_8^4 + \cdots$$
$$a_5^4 = 0 g_4^4 + \alpha_5^4 g_5^4 + 0 g_6^4 + \gamma_5^4 g_7^4 + 0 g_8^4 + 0 + \cdots$$
$$a_6^4 = \beta_6^4 g_4^4 + 0 g_5^4 + \alpha_6^4 g_6^4 + 0 g_7^4 + \gamma_6^4 g_8^4 + 0 + 0 \cdots$$

so that: $\mathbf{a}^4 = (a_4^4, a_5^4, a_6^4)$, $\mathbf{g}^4 = (g_4^4, g_5^4, g_6^4)$, and

$$K^4 = \begin{bmatrix} \alpha_4^4 & 0 & \gamma_4^4 \\ 0 & \alpha_5^4 & 0 \\ \beta_6^4 & 0 & \alpha_6^4 \end{bmatrix}$$

It is clear how we are relying on g_7^4 and g_8^4 to be small for the above procedure to work. We are here helped considerably by the fact that γ_n^m is very small for large n. It contains the factor $(r_0/r_s)^{2n+5}$ which is 2^{-19} for $r_s = 2r_0$ and $n=7$ while α_n^m and β_n^m are still of magnitudes comparable with n.

We shall now attend further details of the computation. The Neumann form of the associated Legendre functions is given by

$$P_{n,m}(\theta) = \frac{(2n)!}{2^n n!(n-m)!} \sin^m\theta \left\{ \cos^{n-m}\theta - \frac{(n-m)(n-m-1)}{2(2n-1)} \cos^{n-m-2}\theta \right.$$

$$\left. + \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2\cdot4\cdot(2n-1)(2n-3)} \cos^{n-m-4}\theta - \dots \right\}$$

When $(n-m)$ is even, the last term inside the curly brackets is a constant, otherwise it is a multiple of $\cos\theta$. The functions $P_{n,m}$ are of very diverse order of magnitude; for example, the mean values of the squares of $P_{4,1}$ and $P_{4,4}$ are in the ratio 1:2016. The functions $P_{n,m}$ will therefore be replaced by other functions, P_n^m, which are merely numerical multiples of $P_{n,m}$ but have the desirable property that the surface harmonics $P_n^m(\theta)\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}m\varphi$ have the same order of magnitude as the zonal functions (i.e. $m=0$) of the same degree. These new functions are not completely normalized in the sense that the average square value of $P_n^m(\theta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\varphi$ over the sphere is unity – it is by the way $1/(2n+1)$ – but they are more convenient in practical work because the coefficients g_n^m and h_n^m indicate at once the approximate orders of magnitude of the corresponding terms in the expansion of the potential that we have developed.

The necessary transformation is given by

$$P_n^m(\theta) = \left\{ q_m \frac{(n-m)!}{(n+m)!} \right\}^{1/2} P_{n,m}(\theta)$$

where $q_m = 2$ for $m > 0$ and $q_m = 1$ for $m = 0$.

Now, every spherical function $P_n^m(\theta)$ has $(n-m)$ real, different zeros <u>between</u> $\theta = 0$ and $\theta = \pi$, so the surface harmonics $P_n^m(\theta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}m\varphi$ vanish along $(n-m)$ circles of latitude, and also, because of the $\cos m\varphi$ or $\sin m\varphi$ factors, along $2m$ meridians at equal intervals π/m. All these zero-lines divide the surface of the sphere into four-cornered (or, at the poles, three-cornered) regions in each of which the sign of the harmonic is constant, while it reverses on crossing the boundary between adjacent regions. Consequently, for $n > m > 0$, the surface harmonics are called tesseral (from tesserae $\equiv$ quadrilateral tablets) harmonics. When $m = 0$, the functions are called zonal harmonics, while the functions are known as sectoral harmonics when $n = m$ because there are no circles of latitude along which the functions vanish.

We introduce now a function W_n^m given by

$$W_n^m = \left\{ \frac{q_m [(2n-1)!!]^2}{(n-m)!(n+m)!} \right\}^{1/2}$$

where we have defined $(2n-1)!! = 1\cdot 3\cdot 5\cdots(2n-3)(2n-1)$ so that $(2n)! = 2^n n!(2n-1)!!$; furthermore a quantity $U_i^{n,m}$ is defined by the following recurrence relation:

$$U_0^{n,m} = 1 \;\; ; \;\; U_i^{n,m} = U_{i-2}^{n,m} \frac{(n-m-i+1)(i-2-n+m)}{i(2n-i+1)} \quad \text{for } i>0$$

Then we can express P_n^m as follows

$$P_n^m(\theta) = W_n^m \sin^m\theta \sum_{i=0,2,4,\ldots}^{n-m} U_i^{n,m} \cos^{n-m-i}\theta$$

Note that i increases in steps of 2.

One should be careful in the calculation of W_n^m in order to avoid overflow in the numerical evaluation of the large factorials. Rather than the <u>algebraic</u> definition given above we would like to give an <u>algorithmic</u> definition reflecting a certain care in handling the numbers:

```
real procedure W(n,m); value n,m; integer n,m;
begin integer k,p,q,r,s; real t;
    r:= 2*n-1; p:= n+m; q:= n-m; s:= 0;
    t:= if m=0 then 1 else 2;
    for k:= 2*n step -1 until 1 do
    begin  t:= t*r; r:= r-s; s:= 2-s;
                if p>1 then begin t:= t/p; p:=p-1 end
        else if q>1 then begin t:= t/q; q:=q-1 end;
    end;
    W:= sqrt(t);
end;
```

If absolutely necessary the square root may be extracted in the inner loop instead. This will slow down the procedure but also improve its numerical robustness.

It can be argued that an algorithmic definition tends to obscure the mathematical clarity and is therefore inferior to an algebraic definition. This is not always the case, however. The following algorithm for computing P_n^m is considerably clearer than the algebraic definition of P_n^m – in addition to being more useful:

```
real procedure P(n,m,theta); value m,n,theta;
integer n,m; real theta;
begin integer i,nm; real s,t,c,c2;
    c:= cos(theta); c2:= c**2; nm:= n-m; s:=t:=1;
    for i:= 2 step 2 until nm do
    begin t:= t*(i-nm-2)*(nm-i+1)/(2*n-i+1)/i;
          s:= s*c2 + t;
    end; comment: now, if nm is odd then... ;
    if nm mod 2 = 1 then s:=s*c;
    P:= s* W(n,m) * sin(theta)**m;
end;
```

Rather than using a library procedure in order to solve the system of linear equations of g^m and h^m we will take advantage of the fact that the matrix K^m is very well conditioned and also that it is rather sparse. The following procedure solves the system efficiently and for g^m and h^m at the same time.

```
procedure solve (K, m, T, a, b); value m,T;
real array K,a,b ; integer m,T;
begin real div, ratio ; integer i,j,h;
  for i:= m step 1 until T do
  begin div := K(i,i);
    for j:= m step 1 until T do K(i,j):= K(i,j)/div;
    a(i) := a(i)/div; b(i) := b(i)/div;
    for j:= m step 1 until T do
    if i<>j then
    begin ratio:= K(j,i);
      if ratio <> 0 then
      begin for h:= m step 1 until T do
              K(j,h):= K(j,h) - ratio * K(i,h);
              a(j):= a(j)- ratio* a(i);
              b(j):= b(j)-ratio* b(i);
      end
    end
  end;
end;
```

Here a and b are the vectors $\mathbf{a}^m$ and $\mathbf{b}^m$. The solution g^m and h^m is returned in place of a and b. Note that the matrix K^m is declared K(m:T, m:T) as a and b (and g and h) are declared a,b(m:T). To get the complete set g_n^m and h_n^m we must solve the system for m varying from 0 to T. This involves of course computation of (T+1) matrices K^m and vectors $\mathbf{a}^m, \mathbf{b}^m$.

To compute K^m, we first write for α_n^m:

$$\alpha_n^m = u_n - \frac{(Q_{n+1}^m)^2}{2n+3} - \frac{(Q_n^m)^2}{2n-1}\left(\frac{r_0}{r_s}\right)^{2n+1}$$

$$= \left((n+1) - \frac{(Q_{n+1}^m)^2}{2n+3}\right) + \left(n - \frac{(Q_n^m)^2}{2n-1}\right)\left(\frac{r_0}{r_s}\right)^{2n+1}$$

$$= \frac{(n+1)(n+2)+m^2}{2n+3} + \frac{(n-1)n+m^2}{2n-1}\left(\frac{r_0}{r_s}\right)^{2n+1}$$

for β_n^m and γ_n^m

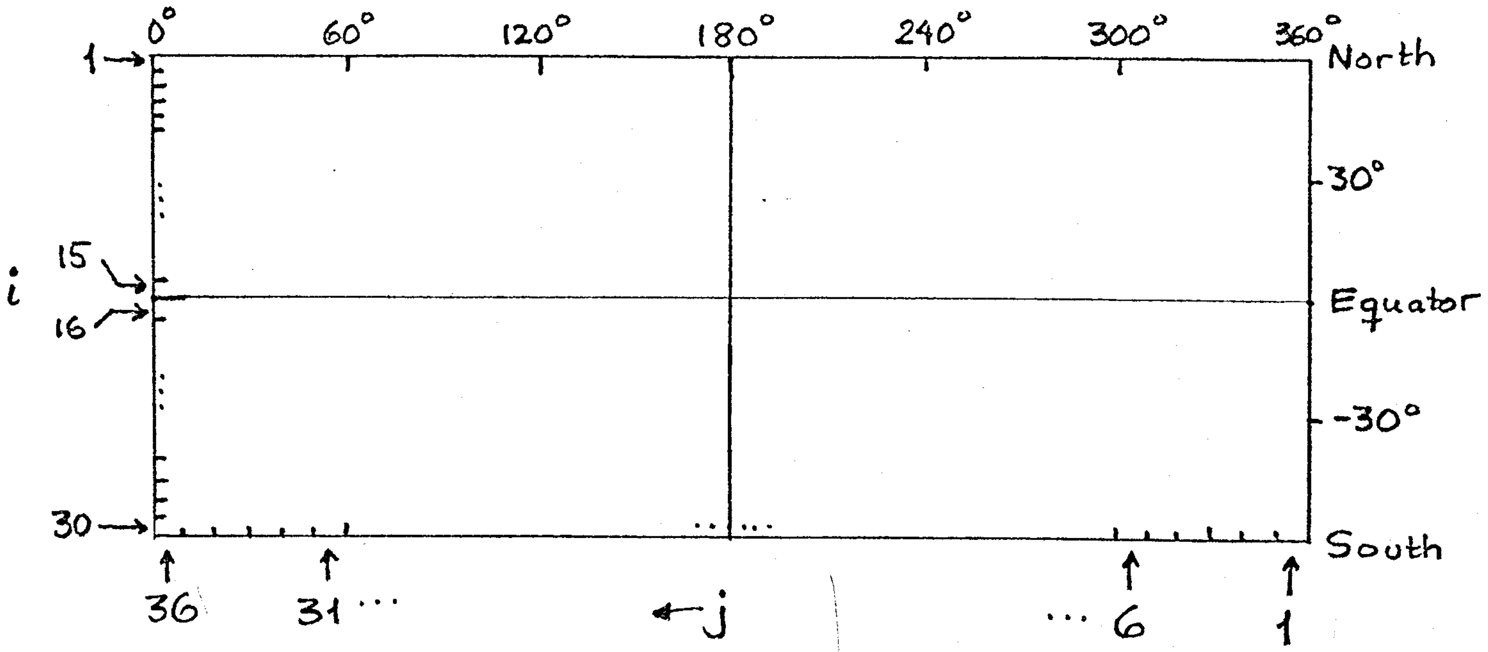
$$\beta_n^m = -\frac{\{((n-1)^2-m^2)(n^2-m^2)\}^{1/2}}{2n-1}$$

$$\gamma_n^m = -\frac{\{((n+1)^2-m^2)((n+2)^2-m^2)\}^{1/2}}{2n+3}\left(\frac{r_0}{r_s}\right)^{2n+5}$$

We set $r_0 = 1$ so that r_s will be expressed in solar radii. The main problem is now to identify the three diagonals of non-zero values of the elements of K^m in terms of array-indices. The following procedure computes the elements of K^m for a given value of the source surface radius r_s. It is called with values of m and T and constructs the K^m-matrix to be used in a call of 'solve'. The computation of K^m could have been performed explicitly inside 'solve', but then we would be violating a sound principle of programming: "every subunit of your computation should be less than a page of text long".

```
procedure getK(K,m,T,rs); value m,T,rs;
real array K; integer m,T; real rs;
begin integer i,n; real m2;  m2:= m**2;
  for n:= m step 1 until T do
  begin for i:= m step 1 until T do K(i,n):= 0;
     K(n,n):= ((n+1)*(n+2)+m2)/(2*n+3)
               + ((n-1)*n + m2)/(2n-1)/rs**(2*n+1);
  end;
  for n:= m+2 step 1 until T do
  K(n,n-2):= - sqrt(((n-1)**2-m2)*(n**2-m2))/(2*n-1);
  for n:= T-2 step -1 until m do
  K(n,n+2):= - sqrt(((n+1)**2-m2)*((n+2)**2-m2))/(2*n+3)
                /rs**(2*n+5);
end;
```

We must now turn to calculation of the vectors a and b. Since we shall be dealing with real data - as real as solar magnetograph data are - we will have to be specific. The magnetograph data is first organized into <u>synoptic charts</u> - using as far as possible observations near Central Meridian only (so that B_φ projected onto the line-of-sight is zero). The sphere is divided into north-south strips each $10°$ wide in heliographic longitude and centered on longitudes $5°, 15°, \ldots, 355°$. We have 36 such strips. A further division into 30 east-west zones is made in sine of heliographic latitude. The customary numbering scheme is shown below:



One may note that the spatial resolution is not the same in longitude as in latitude – the latter having the highest resolution. We are trying to accommodate the fact that solar magnetic fields on a small scale often show considerable zonal organization. How well we actually can do that ultimately depends on the aperture of the magnetograph. With the 36×30 grid we can at most extend the computations to principal index $T \leqq 23$ for the expansion of the potential.

With $N = 30$ and $M = 36$ we have the following mapping:

$$\theta_i = \frac{\pi}{2} - \arcsin\left(\frac{N+1-2i}{N}\right)$$

$$\varphi_j = \frac{2\pi}{M}\left(M - j + 0.5\right)$$

which is easily verified, e.g. by computing θ_1 and θ_{30}, φ_1 and φ_{36}.

Two procedures to compute cosine and sine of θ and $m\varphi$ can now immediately be written down:

```
procedure gettheta(cth, sth); real array cth, sth;
begin real th; integer i;
  for i:= 1 step 1 until ii do
  begin th:= 1.57079637 - arcsin((ii+1-2*i)/ii);
        cth(i):= cos(th);  sth(i):= sin(th);
  end
end;

procedure getmphi(m, cmphi, smphi); value m;
integer m; real array cmphi, smphi;
begin real s, mphi; integer j;
  s:= 2*3.14159265*m/jj;
  for j:= 1 step 1 until jj do
  begin mphi:= (jj-j+0.5)*s;
     cmphi(j):= cos(mphi); smphi(j):= sin(mphi);
  end;
end;
```

Note that we have used global names ii and jj for N and M respectively.

In applying the orthogonality relation to get a_n^m and b_n^m we compute the quantity

$$B_\ell(\theta, \varphi) \sin\theta \, P_n^m(\theta)$$

Rather than computing P_n^m we will define the quantity $X_n^m(\theta)$ by

$$X_n^m(\theta) = \sin\theta \, P_n^m(\theta)$$

or
$$X_n^m(\theta) = W_n^m \sin^{m+1}\theta \sum_{i=0,2,4,\dots}^{n-m} U_i^{n,m} \cos^{n-m-i}\theta$$

and device a procedure to compute X for given n and m for all ii values of θ:

```
procedure getx(x, n,m, sm); value n,m;
real array x, sm; integer n,m;
begin integer i,nm,k; real c,c2,s,u;
  for i:= 1 step 1 until ii do
  begin c:= cth(i); c2:= c**2; nm:= n-m;
    s:=u:=1;
    for k:= 2 step 2 until nm do
    begin u:= u*(k-nm-2)*(nm-k+1)/(2*n+1-k)/k;
          s:= s*c2+u;
    end;
    if nm mod 2 = 1 then s:=s*c;
    x(i):= sm(i)*s*W(n,m);
  end
end;
```

Note that cth (1: ii) is an external or global array holding cos θ and sm (1: ii) is an array holding $\sin^{m+1}\theta$. These arrays are of course introduced in order to avoid recomputing the trig-functions all the time.

With these tools we can write the procedure to compute the vectors $\mathbf{a}^m$ and $\mathbf{b}^m$:

```
procedure getab (BLS, m, T, a, b); value m, T;
real array BLS, a, b; integer a, b;
begin integer i, j, n; real sa, sb, si;
real array x, sm (1:ii), c, s (1:jj);
   for i:=1 step 1 until ii do sm(i):= sth(i)**(m+1);
   getmphi (m, c, s);
   for n:= m step 1 until T do
   begin sa:= sb:= 0;  getx(x, n, m, sm);
    for j:=1 step 1 until jj do
    begin si:= 0;
       for i:= 1 step 1 until ii do si:= si+ BLS(j, i)*x(i);
       sa:= sa + si*c(j); sb:= sb + si*s(j);
    end;
    a(n):= sa*(2*n+1)/(ii*jj);
    b(n):= sb*(2*n+1)/(ii*jj);
   end n;
end;
```

The array BLS(1:jj, 1:ii) is the observed line-of-sight magnetic field. Also, sth(1:ii) just holds $\sin\theta$ and is global to the procedure. The bulk of the running time will be spent in the summation into 'si' i.e. in the loop:

```
for i:=1 step 1 until ii do si:=si + BLS(j, i)*x(i)
```

This is the only place where further optimization will have any significant effect. In an actual implementation of the procedure this loop may with considerable saving in run-time be replaced with an assembler-coded routine.

We are finally in the position to calculate the complete set of coefficients g_n^m and h_n^m up to principal index T:

```
procedure getgh (BLS, rs, T, g, h); value rs, T;
real array BLS, g, h; real rs; integer T;
begin integer m;
  for m:= 0 step 1 until T do
  begin integer n; real array a, b (m:T), K (m:T, m:T);
     getab (BLS, m, T, a, b);
     getK (K, m, T, rs);
     solve (K, m, T, a, b);

     for n:= m step 1 until T do
     begin   g (n,m) := a(n);
             h (n, m) := b(n);
     end;
  end;
end;
```

Before this procedure is called the various global arrays should have been initialized first. The total calculation can be done by a program of about one hundred lines of code. Consequently it is possible to comprehend the program and even gain considerable faith in its correctness.