Terrestrial Magnetism
and
Atmospheric Electricity

VOLUME 36 JUNE, 1931 No. 2

A NEW THEORY OF MAGNETIC STORMS

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PART I—THE INITIAL PHASE

1—*Introduction and summary*

1.1—Many attempts have been made, but hitherto without success, to explain how magnetic storms are produced. The present further attempt is described with a due sense of the pitfalls that abound in this difficult field of speculation. Possibly the fate in store for our theory is only to warn future theorists against some fallacy into which we have unwittingly fallen; yet if so, our work, and that of our critics, may be of value to later writers, just as we have benefited by the labours of past speculators and their critics. But our theory would of course not have been put forward without some confidence on our part in its substantial truth.

1.2—On good grounds, almost every theory of storms* has ascribed them to the action of something propagated to the Earth from the Sun. Lord Kelvin[1] in 1892 showed that the storms could not be directly due to variations in the Sun's magnetic field, and Hale's[2] subsequent measurements of the Sun's field confirm this. The postulated solar agent has therefore been either some corpuscular emission, or ultraviolet radiation.

1.3—"Corpuscular" theories differ as regards the nature and speed of the solar particles, and as to their mode of action when they approach the Earth. The particles are supposed to be electrical, and in the earlier theories, wholly or mainly of one sign; more recently the streams have been thought of as containing charges of both signs, and nearly neutral.

At first storms were regarded as the direct manifestation of the magnetic field of an electric current consisting of a stream of moving

*"Storm" or "field" will throughout this paper mean magnetic storm and magnetic field, unless otherwise indicated.

[1]Proc. R. Soc., 52, 307-308 (1892).

[2]Astroph. J., 38, 27-98, 99-125 (1913); 47, 206-254 (1918). [Contrib. Mt. Wilson Obs., Nos. 71, 72, and 148.]

electric charges of like sign. Lodge,[3] in 1909, proposed this view, suggesting that the particles were electrons; he supposed them to travel past the Earth in a straight stream (though such a stream could not really explain the known characteristics of a storm).

Birkeland[4] in 1896 pointed out that electrons traveling toward the Earth would be deflected by the Earth's field. He made experiments in which cathode-rays were projected towards a small magnetized sphere, and showed that their distribution near the sphere was largely governed by the field, which, in particular, guided many of the electrons towards the polar regions. This gave the first insight into the physical cause both of the special intensity of magnetic disturbance in high latitudes, and of the polar incidence of aurorae. Since then the theories of the two phenomena have been linked together. Birkeland[5] developed his ideas in three extensive and important memoirs, of 1901, 1908, and 1913; these form the most detailed and comprehensive theoretical discussion of magnetic storms hitherto published. His work has been continued and developed in Norway by Störmer[6] and Vegard,[7] but with particular reference to auroræ rather than to magnetic storms. Birkeland supposed the corpuscles that are responsible for aurorae and magnetic storms to be electrons of very high velocity, nearly equal to that of light. Störmer and Vegard have considered also the possibility that they may be slower electrons or (positive) α-particles.

Schuster[8] criticized storm-theories of the type proposed by Lodge and Birkeland, on the ground (among others) that, without unreasonably high velocity and energy, the stream would not hold together against the mutual repulsion of its parts during the passage from the Sun to the Earth. He regarded Birkeland's *auroral* theory, however, as still tenable (§ 1.31), if the electron-stream was suitably rare (too rare to produce an appreciable direct magnetic field).

1.31—Theories of the type considered in § 1.3 attribute the storm to electric currents flowing mainly in the space outside the Earth's atmosphere—the region being, indeed, many times larger than the Earth in linear extent. An alternative totally different view is that the currents which produce the storm flow wholly in the Earth's atmosphere. This view was suggested by the dynamo-theory of the daily magnetic variations, originated by Balfour Stewart[9] and developed quantitatively by Schuster in 1889[10] and 1905.[11] In reference to magnetic disturbance this view seems to have been first proposed by Ad. Schmidt[12] (1899), who regarded the typical "element" of magnetic disturbance as being due to a moving current vortex in the upper atmosphere. The same idea seems to have been favored, though not definitively, by van Bemmelen[13] (1900), relative to the simple world-wide component of the

[3]Nature, **81**, 425-426 (1909).

[4]Arch. Sci. Phys., Genève, **4**, 497 (1896).

[5]Skr. Vid. selsk., No. 1 (1901); Norwegian Aurora Polaris Expedition, **1**, Sect. 1 (1908), Sect. 2 (1913).

[6]Terr. Mag., **35**, 193-208 (1930) and references there given.

[7]Wien-Harms, Handbuch der Experimentalphysik, **25**, 385-476 (1928) and earlier references there given.

[8]Proc. R. Soc., A, **85**, 45-50 (1911).

[9]Encyclopaedia Britannica, 9th ed., **16**, 181-184 (1878).

[10]Phil. Trans. R. Soc., A, **180**, 467-518 (1889).

[11]Phil. Trans. R. Soc., A, **208**, 163-204 (1898). See also S. Chapman, ibidem. **213**, 279-321 (1913); **218**, 1-118 (1919).

[12]Met. Zs., **16**, 385-397 (1899).

[13]Terr. Mag., **5**, 123-126 (1900).

storm-field corresponding to the diminution of horizontal force during the main phase of the storm. These writers expressed no views as to how the currents were set up. In 1905, and again in 1911, Schuster advocated the same "atmospheric-current" hypothesis; he amplified it by proposing that the currents are impelled by electromotive forces which are always present in the upper atmosphere, their effect, however, being magnified from time to time by increased ionization and conductivity of the air, due to the entry and impact of solar corpuscles. He considered that the energy of the storm is drawn from the kinetic energy of the Earth's rotation; he did not explain or discuss the detailed characteristics of the storm-field.

In the concluding part of a paper[14] devoted mainly to an observational analysis of storms, Chapman made the first detailed attempt to develop a storm-theory of the atmospheric-current type. His aim was to explain the very definite characteristics of storms brought out by his analysis of them, following that of Moos.[15] He ascribed the currents to the action of a stream of corpuscles mainly of one sign of charge. While qualitatively successful within certain limits, the theory, on further examination, proved to have many defects, and it was afterwards given up by its author; for a time it was still upheld by Angenheister,[16] with the modification that horizontal motion was supposed to accompany the main, vertical, air-motion considered in the theory.

Lindemann[17] in 1919 criticized Chapman's numerical development of his theory, chiefly on the ground that it involved an accumulation of charge in the Earth's atmosphere which would, by electrostatic repulsion, prevent the supposed continuous entry of further charges.

In his first paper Chapman did not consider the storm-data derived from polar stations, though proposing to do so in a later paper[18]—an intention fulfilled in 1927. The new features then brought out were inconsistent with his dynamo-theory of storms. While abandoning this particular theory, he still favored the atmospheric-current hypothesis, though unable to construct a theory of this type which would fit the observed facts.

1.32—Lindemann added to his criticism of Chapman's theory a suggestion that the latter might be preserved in substance, if the supposed stream of charged particles, mainly of one sign, were replaced by a neutral but ionized stream or cloud. On entry into the Earth's atmosphere the electrons would be stopped at a higher level than the more massive positive ions, and a vertical separation of charge would result, which he took to be the essential requirement in Chapman's theory.

He showed how a neutral ionized stream might originate at the Sun and travel to the Earth, without appreciable recombination, and with a velocity of about 8×10^7 cm/sec (or 800 km/sec). He did not consider its terrestrial effects, though he suggested that it might produce aurorae as well as storms.

In 1923 Chapman[19] examined whether such an ionized stream, strictly neutral electrostatically, would be deflected in the Earth's magnetic field so as to impinge upon the atmosphere mainly in the polar regions.

[14]Proc. R. Soc., A, **95**, 61-83 (1918).
[15]Colaba Magnetic Data, **2**, Chap. 10 (1910).
[16]Göttingen, Nachr. Ges. Wiss., 1-42 (1924).
[17]Phil. Mag., **38**, 669-684 (1919).
[18]Proc. R. Soc., A, **115**, 242-267 (1927).
[19]Cambridge, Proc. Phil. Soc., **21**, 577-594 (1923).

He concluded that it would be deflected only very slightly, and not in such a way as to produce the aurora polaris. It may also be noted (though this was not stated in Chapman's paper) that, contrary to Lindemann's supposition, the entry of the particles into the atmosphere would not produce the effects ascribed in Chapman's theory to a set of charges mainly of like sign. No alternative explanation of storms appeared to result from the motion of the neutral ionized stream near the Earth.

1.4—Recently (1929) Hulburt and Maris[20] have endeavoured to explain the main facts[17, 18] about storms, and also about aurorae, by the hypothesis that these phenomena are produced by *terrestrial* corpuscles, temporarily separated from the Earth's atmosphere, to a distance of about five earth-radii, by the action of solar ultraviolet light. While their theory is ingenious and interesting, according to Chapman[21] it does not in reality succeed in explaining magnetic storms.

1.5—During the past three years we have made repeated attempts to construct a satisfactory corpuscular theory of storms. The tendency for magnetic disturbance to recur at intervals of a solar rotation, discussed and interpreted by Maunder,[22] and further confirmed by Chree,[23] seemed to us to afford a strong presumption that, in some way, streams of solar corpuscles were the cause of storms. The interesting work of Milne[24] on the emission of high-speed corpuscles from the Sun further encouraged this view. A re-examination[25] of the conditions of passage of the stream from the Sun to the Earth disposed of our lingering hope that the stream might carry some small residue of charge which would at least suffice[26] to explain the production of aurorae, by permitting the stream to be deflected in the same way as separate corpuscles are in Störmer's theory (though to a smaller extent); our work confirmed Lindemann's conclusion that the only admissible kind of stream is one that is electro-statically neutral to a very high degree of approximation.

After many unsuccessful endeavours to find how such a stream could produce a storm when near the Earth, we returned to consider whether, as Chapman[27] suggested in 1920, the Sun could emit streams which, though having no space-charge, carry an electric current by reason of a difference between the streaming velocities of the positive ions and the electrons. According to Milne's emission-theory, the positive ions are projected outwards by selective radiation-pressure, while the radiative acceleration of the electrons is much smaller. The electrons must be drawn after the ions by electrostatic forces, but it seemed possible that they might follow the ions at a somewhat slower rate, so that, while the number of each per unit volume was the same, and the space-charge consequently zero, the stream would convey a positive current. An ideal problem bearing on this question was devised by us, and solved by Ferraro;[28] the result convinced us that the stream could not carry any appreciable current, or possess an appreciable magnetic field. Thus

[20] Phys. Rev. **33**, 412-431 (1929); **34**, 344-351 (1929); **36**, 1560-1569 (1930).
[21] Mon. Not. R. Astr. Soc., Geophys. Sup., **2**, 296-300 (1930).
[22] Mon. Not. R. Astr. Soc., **65**, 2-35, 538-559, 666-681 (1904).
[23] Phil. Trans. R. Soc., A, **212**, 75-116 (1912); **213**, 245-277 (1913).
[24] Mon. Not. R. Astr. Soc., **86**, 459-473 (1926).
[25] Chapman and Ferraro, Mon. Not. R. Astr. Soc., **89**, 470-479 (1929).
[26] Q. J. R. Met. Soc., **52**, 225-236 (1926).
[27] Phil. Mag., **40**, 665-669 (1920).
[28] Mon. Not. R. Astr. Soc., **91**, 174 (1930).

the only kind of stream available for a corpuscular theory of storms was one of the kind proposed by Lindemann, neutral, ionized, and with the same streaming velocity for both the ions and electrons.

1.6—If magnetic storms are due to solar corpuscles, as still seemed to us likely, it was therefore necessary to return once more to the problem of the motion of such a stream near the Earth, despite the earlier failure to find how the motion could explain a storm.

The properties to be assigned to the streams were necessarily largely hypothetical, since as yet we have no direct evidence even as to their existence. A possible method of testing the latter, and, if successful, of determining the density of the stream, has been suggested by Chapman,[29] but not yet tried; considerations relating to the solar atmosphere, however, render it likely that at a distance from the Sun equal to the radius of the orbit of the Earth, the density (apart from any change brought about by a deflecting influence of the Earth's magnetic field) will lie between 2×10^9 and 20 ions/cc.

The work of Lindemann[17] and Milne[24] suggests that the streaming velocity will be at least of the order 10^8 cm/sec (1,000 km/sec); distinctly higher velocities seem possible, indeed, since storms appear to be associated with disturbed areas on the Sun, where higher temperatures and greater radiation-pressure may operate. The geometry of the streams, for any given velocity, taking account of the solar rotation, has been worked out;[29] though the individual particles travel almost radially outwards from the Sun, the outline of the stream rotates with the Sun; a continuous stream will overtake the Earth in its orbital motion, approaching on the *post meridiem* (P. M.) side; the surface of the stream, if undisturbed by the Earth's magnetic field, would cross the Earth in about 35 seconds.

The radius of cross section of the stream at the distance of the Earth will depend on the area of the emitting region on the Sun, but if it is assumed (with some plausibility) that the duration of a storm is approximately the same as the period during which the Earth is enveloped in the stream, the radius may be estimated as about 10^{12} to 10^{13} cm, or 2×10^3 to 2×10^4 Earth-radii.

According to Milne's emission-theory, the stream may include *neutral* atoms as well as ions and electrons; the atoms and ions would be emitted by the same process, and their numbers may be expected to bear some roughly constant ratio to each other. The neutral particles will of course travel through the Earth's magnetic field without deflection, and impinge on the sunlit hemisphere, there increasing the ionization of the atmosphere by impact with the air-molecules. This excess ionization may indirectly affect the development of the magnetic storm, but the neutral atoms are unlikely to play any direct or essential part in producing the storm.

1.61—If the stream is composed of N positive ions and N electrons per cc (N varying from point to point), which have random motions corresponding to a temperature T, taken to be 6,000°, then the mean free-path l of the electrons is about $2\times10^{11}/N$ cm, according to one mode of estimation.[30] Another method is to suppose that a collision (terminating a free-path) occurs if a particle suffers a deflection of 90° or more: in a rare stream—such as will be considered here—this deflection will be produced mainly by the cumulative effect of a series of feeble

[29]Mon. Not. R. Astr. Soc., 89, 456, 466 (1929).

encounters (rather than by a single encounter).* On calculation, we find very nearly the same value for the mean free-path as that given above by the first method.

The coefficient of diffusion D of the ions and electrons relative to one another is very high.[31] For singly charged ions (of any kind) at 6000°, it is approximately $1.8 \times 10^{18}/N$ $(1 - 0.048 \log N)$. Thus ND is nearly constant over a wide range of values of N; for example, 10^{-18} ND is 3.5, 4.6, and 6.5 for $N = 0.1$, 10^4, and 10^9.

The electric conductivity σ of the gas is proportional to ND, and therefore is nearly the same over a wide range of N. For $ND = 5 \times 10^{18}$, $\sigma = 1.6 \times 10^{-9}$ e.m.u.; this conductivity is considerable, though much less than that of copper (6×10^{-4}) at ordinary temperatures.

If there are neutral atoms in the stream, as well as ions and electrons, the estimates of l, D, and σ must be somewhat reduced, but probably only by a factor of order one-half or one-third.

In the presence of a magnetic field of intensity H, the ions and electrons may be in certain cases spiral in paths of radius R, where $R = m\,v/e\,H$, m and v being the mass and velocity of a particle, and e its charge in e.m.u. For an electron $m\,v/e = 2.83$, and for hydrogen (H^+) and calcium (Ca^+) ions respectively it is 1.26×10^2, 8.06×10^2. When this spiralling occurs the contributions of the electrons and ions to σ are reduced. In a rare stream, in which the *ions* contribute practically the whole of the current, σ is approximately reduced in the ratio of (m_i/m_e) $[R^2/(R^2 + l^2)]$, m_i and m_e denoting respectively the masses of an ion and an electron.

1.7—Our re-examination of the motion of a neutral ionized stream in a magnetic field indicates that Chapman's investigation[19], though defective in some points, is correct as regards one of its main conclusions, namely, that within the stream the ions and electrons can move together nearly rectilinearly, without spiralling,† and with only a slight deflection by the field. The field tends to deflect the ions and electrons differently, and so to separate them; but this tendency is resisted by the electrostatic field thereby set up, and all that results is a slight "polarization" of the stream, involving a surface charge-distribution, and also in general, if the polarization is not uniform, a volume charge-distribution. The electrostatic energy of the polarization is drawn from the kinetic energy of the stream, which is slightly slowed down.

In Chapman's investigation the stream was supposed to envelop the Earth completely, its lateral surface being so far from the Earth that the surface-charge there was negligible. In our work we have carefully examined this and other surface-effects, which were either neglected by him, or did not enter into the problem that he considered, namely, that of a stream in the *steady state*. We find them to be of primary importance, however, in the production of a magnetic storm, which depends essentially on surface-phenomena associated with the approach of the stream towards the Earth. Chapman's failure to recognize the mode of production of the storm is explained by the limitation which he placed on his problem at the outset.

1.8—The surface-effects that we have investigated are of three

*Cf. J. Jeans, Astronomy and Cosmogony, p. 311 (1928).

[30]Mon. Not. R. Astr. Soc., 89, 79 (1928).

[31]Mon. Not. R. Astr. Soc., 82, 292-297 (1922); Persico, Mon. Not. R. Astr. Soc., 86, 93 (1926).

†It is assumed that the streaming velocity is many times larger than the thermal velocities of the ions and electrons.

kinds. The mere presence of the surface-charge, uncomplicated by other effects that occur in general, is exemplified in the case of an infinite-plane-slab stream moving through a uniform magnetic field (§ 2); here the surface-charge is stable, and its sole function is to neutralize the deflecting tendency of the magnetic field on the particles inside the stream.

The second effect occurs in the next simplest case, that of a cylindrical stream of any cross-section moving through a uniform field. Here the surface-charge density is non-uniform, and there is an electric field *outside* the stream as well as within it: the surface-charge is repelled from the surface and tends to escape.

The third effect appears (together with the other two) in the case of a stream moving in a non-uniform field; for definiteness, suppose the stream is advancing into a region of increasing intensity. The polarization and the surface-charge on a particular section of the stream increase as it moves along; the deflecting tendency of the field inside the stream is nearly neutralized, as before, and the surface-charge continuously escapes; but now, in addition, electric currents are induced, in and near the surface of the stream, which tend to maintain the magnetic field uniform inside the stream—that is, to reduce its intensity as the stream advances into the stronger field; conversely the currents increase the field outside the stream. The stream may be thought of as offering resistance to its interpenetration by the tubes of magnetic force, so that it pushes them before it in its motion, increasing the magnetic intensity in front of it, and reducing it within its surface. This third effect is, we think, the cause of the initial phase of a storm, while the second (the escape of surface-charge) is important in producing the main phase.

The magnetic field exerts a mechanical force on the surface-currents, and tends to oppose the advance of the stream into the more intense field. In the case of a stream from the Sun advancing towards the Earth, a hollow space round the Earth is formed in the stream; the hollow is open at the back of the Earth (as viewed from the Sun); this is illustrated in Figure 1. The hollow gradually shrinks, at a diminishing rate as the surface advances; meanwhile the surface-layer is increased in density by the inpouring of the particles coming on from behind, which are not retarded till they enter the current-carrying layer. If the stream were directed towards the Earth for an indefinite period, the hollow would eventually close up on to the Earth, and the conditions would become in some ways similar to those of the problem discussed by Chapman;[19] but by that time the storm (or at least some of its most important phases) would be over. However, owing to the rotation of the Sun the stream may pass on before this occurs; part of the stream which has collected near the Earth remains for a time, probably in the form of a ring round the equator; this gradually disappears by the passage of the ions and electrons, along the Earth's lines of force, into the atmosphere in high latitudes.

2—The motion of an infinite plane-slab stream in a uniform magnetic field

2.1—The velocity u of the stream to be considered will be of the order of 10^8 to 10^9 cm/sec, so that $(u/c)^2$, where c denotes the velocity of light, is of the order 10^{-3} to 10^{-5}; in general we shall neglect terms of the second degree in u/c, as compared with unity.

The magnetic field of the Earth has its maximum intensity at the poles (except for a few limited disturbed localities), where the intensity

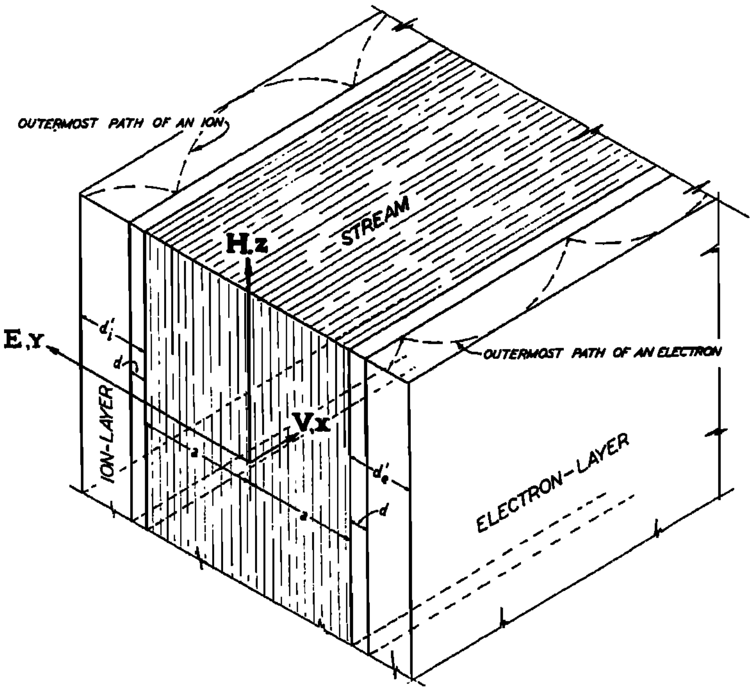

FIG. 1

H is less than one gauss. The fields to be considered will therefore be of this or smaller intensity.

The motion of a single particle of mass me, velocity $\mathbf{v}$ and charge e (e.s.u.) in a uniform electric and magnetic field E, H, is given by the equation*

(1) $$m\dot{\mathbf{v}} = e\mathbf{E} + (e/c)\ \mathbf{v} \wedge \mathbf{H},$$

the solution of which is, if the direction of $\mathbf{E}$ and $\mathbf{H}$ form a right-angle

(2) $$v = c(\mathbf{E} \wedge \mathbf{H})/H^2 + \mathbf{v}_0;$$

$\mathbf{v}_0$ is a vector, the value of which can be chosen arbitrarily at a particular instant, and which thereafter satisfies the equation

(3) $$\dot{\mathbf{v}}_0 = (e/mc)\ \mathbf{v}_0 \wedge \mathbf{H};$$

thus the component of $\mathbf{v}_0$ parallel to $\mathbf{H}$ is constant, while the normal component rotates about the direction of $\mathbf{H}$ with the angular velocity $\omega = eH/mc$. The motion is therefore trochoidal, and the radius R of the circular component of the motion is v_0/ω or mcv_0/eH (taken positive whatever the sign of e).

*The notation $\mathbf{a} \wedge \mathbf{b}$ is used for the vector-products of two vectors, $\mathbf{a}$, $\mathbf{b}$.

If $\mathbf{E} = E\mathbf{y}$, $\mathbf{H} = H\mathbf{z}$, (2) becomes

(4) $$\mathbf{v} = (cE/H)\,\mathbf{x} + \mathbf{v}_0,$$

where $\mathbf{x}$ is the unit vector $\mathbf{y}_\wedge\mathbf{z}$; if $\mathbf{v}_0$ has no $\mathbf{z}$ component, the motion is purely in the $\mathbf{x}$, $\mathbf{y}$ plane.

2.2—Consider a uniform "slab" of neutral ionized gas, *at rest*, between two infinite parallel planes $y = \pm a$, in the presence of uniform applied electric and magnetic fields $E\mathbf{y}$, $H\mathbf{z}$ throughout space. The gas can be maintained in equilibrium by electrically charged surface-layers, containing either ions only (at $y > a$) or electrons only ($y < -a$), the charge per unit surface-area being $\sigma = E/4\pi$; between the planes $y = -a$ and $y = a$ the electric field due to these layers neutralizes the applied electric field $E\,\mathbf{y}$.

Let the density of the singly charged positive ions be N per cc, and likewise that of the electrons, not only in the neutral gas between $y = \pm a$, but also initially in the surface-layers; the thickness d of these layers is therefore given by $\sigma = Nde$, where e is the electronic charge (in e.s.u.). Alternatively the density (N_i) of the ions in the positively charged layer, and that (N_e) of the electrons in the negative layer may be different from N and from each other, in which case the thickness of the layers (d_i and d_e, say) will be related by the equations

$$N_i d_i e = N_e d_e e = \sigma$$

The whole set of charges being initially at rest, those in the interior neutral region will remain so, but not those in the surface-layers. In the stratum at the distance θd from the inner surface of either layer the charges will describe cycloidal paths; for initially and (as will appear) always, the electric force on this stratum is $\theta\mathbf{E}$; under the influence of this and the magnetic field H, and subject to the initial conditions $y = \pm(a + \theta d)$, $\mathbf{v} = 0$, the subsequent motion is given by

(5) $$\mathbf{v} = (\theta cE/H)\,\mathbf{x} + \mathbf{v}_0,$$

where initially $\mathbf{v}_0 = -(\theta cE/H)\,\mathbf{x}$; thus the mean velocity and the "circular" velocity are equal, so that the path is a cycloid; the radius R of the circular motion is $\theta mc^2E/eH^2$, and the y-coordinate at any time t is given by

(6) $$y = \pm[a + \theta\{d + (1\text{-}\cos\omega t)\,mc^2E/eH^2\}] = \pm(a + \theta d') \text{ say,}$$

where $\omega = eH/mc$. The thickness of the layer at time t is d', as defined by (6); it varies periodically in the time $2\pi/\omega$. From (6) it is clear that the layer expands and contracts uniformly, each stratum dividing the layer in the same ratio θ throughout. Thus the electric field θE in the layer remains constant, as stated. The density Nd/d' of each layer remains uniform if it was so initially, but varies periodically. Besides the lateral expansion and contraction, the layer undergoes a longitudinal shearing motion, which varies periodically; the mean rate of shear is E/cH.

The values of d' and $2\pi/\omega$ are different for the two layers, being in the ratio of the masses of the charges. Thus the ionic layer pulsates with far greater amplitude, and much more slowly, than the electronic layer. Also the weaker the magnetic field H, the slower will be the pulsations of the layers and the greater their amplitudes.

2.3—If the slab, while possessing no mean motion, is a gas at temperature T, the particles within the neutral interior will spiral about the lines of magnetic force; the component of their motion transverse to $\mathbf{z}$

will rotate about **z** with the angular velocity ω, while their **Z**-motion will remain constant except so far as it is changed by collisions. The charges near the planes $y = \pm a$, which are carried by their spiral motion into the charged layers, will be affected slightly by the electric field in these layers; and if the particles of the layers themselves also have random motion, the conditions in the two layers will not be quite so simple as those indicated above. But the layers will be stable, since any charge which, by its random motion, is carried beyond its fellows out of the layer, will be restored to the layer by the deflecting action of the field. The random motions slightly blur the simple conditions previously described, but without altering their character in any essential feature.

2.4—Next consider the slab with respect to axes in uniform motion relative to it, with the velocity $-u\mathbf{x}$, so that the slab itself becomes a slab-stream having the velocity $u\mathbf{x}$ relative to these axes. If u^2/c^2 is negligible compared with unity, as we suppose, the new values of the electric and magnetic intensities at a point where in the former case they were $\theta E\,\mathbf{y}$, $N\mathbf{z}$, now become

$$(\theta E + uH/c)\mathbf{y},\ (H + u\theta E/c)\mathbf{z}$$

Suppose that

(7) $$u/c = -E/H$$

so that E must be considered small in comparison with H; then since in the former case the electric field was $E\mathbf{y}$ (corresponding to $\theta = 1$) everywhere outside the slab, it is now everywhere zero outside the slab-stream; inside the slab it was zero (corresponding to $\theta = 0$), so that inside the slab-stream it is $(uH/c)\mathbf{y}$. In the surface layers it varies uniformly from zero at the outer boundary to $(uH/c)\mathbf{y}$ at the inner one, being $(1-\theta)\ (uH/c)\mathbf{y}$ for the stratum θ. The magnetic field is everywhere approximately $H\,\mathbf{z}$, because the term $u\theta E/c$, which is zero outside the stream, $-(u^2/c^2)\theta H$ in the charged layers, and $-(u^2/c^2)H$ within the stream, is negligible in comparison with H; in fact, we have already neglected variations of **H** of this order by not considering the magnetic effect of the electric currents corresponding to the shearing motion of the layer. This will be considered in § 2.51.

2.5—Thus the former problem, referred to the new axes, gives the approximate solution for a uniform slab-stream moving with velocity $u\mathbf{x}$ in the presence of a magnetic field $H\mathbf{z}$, and in the absence of any external electric field. The slab will be polarized, having two surface-layers containing only charges of one sign; these are initially of thickness

(8) $$d = uH/4\pi Nce$$

but pulsate so that at time t their thickness d' is given by

(9) $$d' = d + (1-\cos\omega t)\ (mcu/eH),$$

where mcu/eH is taken positive whatever the sign of e; the paths of the particles of the layer are now trochoidal, the mean velocity in the stratum θ being $(1-\theta)u$, and the velocity of circular motion θu. Inside the slab, in the neutral region, the particles execute straight paths with velocity u, subject to the balanced forces $(uH/c)\ \mathbf{y}$ (electric) and $(-\ uH/c)\ \mathbf{y}$ (electromagnetic).

2.51—The variation of H, which, as already indicated in § 2.4, is only of the second order in u/c, will be oscillatory, since, in addition to their mean velocity $(1-\theta)u$ ($\equiv V$ say), the particles are endowed with circular motion: the mean value of the variations of H may be obtained readily from the equations satisfied in the charged layers, where (alone) there is an electric current.

These equations are:

$$\delta H/\delta y = 4\pi NeV/c, \ \delta E/\delta y = 4\pi Ne, \ E = VH/c,$$

hence,

$$H\delta H/\delta y = 4\pi NeVH/c = E \ \delta E/\delta y$$

or,

$$\delta(H^2-E^2)/\delta y = 0$$

Thus, if H_0 is the value of H outside the stream (where $E = 0$) in the charged layers and within the stream $H^2 = H_0^2 + E^2$. Within the stream, therefore, $H^2 = H_0^2 + V^2H^2/c^2$ or $H = H_0 \ (1-V^2/c^2)^{-1/2}$.

2.6 – If the gas is at a temperature of 6000°, as will be supposed in the subsequent work, the particles will possess random motions relative to the mean velocity of the stream; let V' denote the mean thermal velocity of the particles; V' for Ca-atoms, H-atoms, and electrons at a temperature of 6000°, is approximately 2×10^5, 10^6, and 5×10^7 cm/sec respectively. All of these are smaller than the values of u to be considered (10^8 to 10^9 cm/sec).

Corresponding to the spiral motions inside the stationary slab (§ 2.3), the internal particles of the slab-stream will have like spiral motions superposed on $u\mathbf{x}$; in the $\mathbf{x}$ direction the motion will always be forward, but fluctuating; there will also be a fluctuating transverse motion parallel to $\mathbf{y}$, and a random transverse motion parallel to $\mathbf{z}$ (steady except for collisions). These velocities superposed on $u\mathbf{x}$ are very slight compared with u for any kind of ions, and for them the velocity-vector will never deviate appreciably from the $\mathbf{x}$ direction; for the electrons the velocity-vector may be inclined considerably to the $\mathbf{x}$ direction, and steadily so (except for collisions) in the xz-plane. The maximum deviation of the paths from their mean rectilinear course (a straight line in the xz-plane) varies between $+ mcV'/eH$, which for $H = 1$ gauss, is only 3 cm for the electrons, and about 800 cm for Ca-atoms. These deviations may be compared with the corresponding ranges of pulsation of the charged surface-layers, which are mu/eH, that is for $H = 1$, about 6 cm for the electrons and 4×10^5 cm (4 km) for Ca-atoms, taking $u = 10^8$ cm/sec.

The random motions of the particles in the surface-layers will modify the ideal conditions described in § 2.5 to a similar degree, and therefore by amounts which, as regards displacements in the y-direction, are absolutely small for the electrons, and relatively small (compared with the thickness of the layer) for the ions. Thus the main features of the solution are unaffected by the random motions.

2.7 – It has been tacitly assumed that the stream has its equilibrium polarization from the outset of the problem. If there is no initial magnetic field in the presence of the slab-stream, and a slowly increasing uniform field is introduced, the polarization will be gradually set up; we have not thought it worth while to attempt to work out this very difficult problem in detail, because it seems clear that the conditions at any time will be approximately, though not exactly, the same as those already described for a constant field equal in strength to the actual field existing at this instant. Initially, the minimum thickness d of the charged layers is small, but the lateral pulsations are large and slow. The latter decrease, and become more rapid as the field increases and, with it, the minimum thickness of the charged layers. If the field increases so slowly that the outer particles in the charged layers have time to get far away from the initial boundary of the stream during the slow initial pulsations, some of them may be left behind there, to spiral in paths of diminishing radius as the intensity of the field increases.

2.8—If the field H is increased somewhat beyond the value

$$H_0 \equiv (4\pi ce/u)\ ND \tag{10}$$

where D is the unpolarized thickness of the stream, the two sets of charges will be completely separated by the field, because the polarization displacement d, as given by (8), will exceed D. The charged layers will include all the ions (or all the electrons) in the stream, with no region of neutral gas between. The maximum electric field which their separation can produce is $E_0 \equiv 4\pi eND$, which is insufficient to balance the electro-magnetic deflecting force on the charges when $H > H_0$, at least if their velocity is $u\mathbf{x}$. Actually, as H increases from zero to H_0, the inner surfaces of the charged layers move laterally, subject to opposing electric forces, and the velocity and kinetic energy of their particles is reduced; thus the maximum electromagnetic force on these particles is less than uH/c, and a correspondingly smaller electric force suffices to balance it; therefore a value of H somewhat exceeding H_0 is necessary for complete separation of the charges. When the charged layers are separated, the mean velocity of the inner surface of each will be cE_0/H or $4\pi ceND/H$ or uH_0/H, while for the outer surface it remains zero as before. The inner surface, as well as the other strata of the layers, will pulsate, owing to the y-component of velocity acquired by the stratum while the polarization is being set up. The two layers will not move right away from one another, but will pulsate side by side, perhaps overlapping one another to some extent at certain stages during their pulsations.

2.9—The stream has been supposed to move transverse to the field, but it may also have a component of velocity parallel to the field. This component will remain constant, and the consequences of the motion transverse to the field, as discussed above, will not be affected.

3—*The motion of a cylindrical stream in a uniform magnetic field*

3.1—The next simplest form of stream to consider is an infinitely long cylinder of circular cross-section, in a uniform magnetic field. For simplicity, we suppose the stream to be already polarized by a displacement

$$d = uH/2\pi Nce \tag{11}$$

(half that in the case of the slab, which we shall assume to be small, either through uH being small or N large). This will produce an electric field that will balance the electromagnetic deflecting force inside the stream. But this case differs essentially from that of the slab in that there is now an electric field outside the stream, and the charged layer will tend to escape rapidly under the influence of this field.

3.2—Take the x- and z-axes, respectively, along the axis of the cylinder and parallel to the magnetic field; the initial surface density of charge at a point $y = a \cos\theta$, $z = a \sin\theta$, where a is the radius of the cylinder, will be $\sigma = -\sigma_0 \cos\theta$, σ_0 being the surface-density Ned, or $uH/2\pi c$ in the "equatorial" plane of the cylinder ($z = 0$). The electric potential outside the cylinder will be $-2\pi\sigma_0(a^2/r)\cos\theta$; the r, θ components of the electric force will be equal to $-2\pi\sigma_0(a/r)^2\cos 2\theta$, $-2\pi\sigma_0(a/r)^2\sin 2\theta$. The latter will accelerate the surface-charges away from the plane $z = 0$, very nearly along the direction of the lines of magnetic force, while the transverse (y) electric force, in conjunction with **H**, will merely cause a fluctuating shearing and pulsating motion in the charged layer. On account of the electric force outside the stream, the mean x-velocity in the charged layer, in the equatorial plane, will vary from $u\mathbf{x}$ on its inner surface to $-u\mathbf{x}$ on the outer one, instead of from $u\mathbf{x}$ to 0, as in the case of the slab.

3.3—The electrons forming the negatively charged layer ($y > 0$) will escape (in the $\pm z$-directions) much more rapidly than the ions, because while the former have much the smaller mass, the accelerating electric force is (at the beginning) the same for them as for the ions. They will therefore travel far beyond the latter, in equal times.

As charges escape from the surface, new charges will flow to it from the interior so as to maintain the internal field at the uniform value $(uH/c)\mathbf{y}$, just sufficient to balance the electromagnetic deflecting force. This corresponds with the fact that, relative to axes moving with the stream, the electric field inside it must be zero (to a high order of approximation), since the stream is a good electrical conductor. The internal field is due to the charges over the surface and those that have escaped from it.

3.4 The rapid escape of electrons from the half of the stream lying in the region $y > 0$ will leave behind, on or near the surface, an excess of positive charge. This will increase, and will slow down the electrons and increase the rate of escape of the ions, tending to equalize the rate of escape of the charges of opposite sign. If the steady state can be attained or approached, the rate of escape of ions and electrons across any plane z = constant ($> a$) must be nearly equal. But, as will appear, the electrons on first leaving the surface have higher velocities than the ions, so that the number of the escaping charges between any two planes $z = z_1$, $z = z_2$ is greater for the ions than for the electrons. Hence the total charge between two such planes is positive except in the most distant regions where the electrons are that escaped first.

3.41—If we could regard the escaped charges as being too far away to affect the electric field inside the stream, this field, which has the value $(uH/c)\mathbf{y}$, would then be due to the surface-charge. The only way in which a uniform field can be produced inside a cylinder when the total surface-charge is not zero, is for the excess charge to be spread uniformly over the surface; the combined density σ of the distribution is then equal to $\sigma_+ - \sigma_0\cos\theta$, where σ_+ is the mean excess charge per unit area. The area over which the charge is negative is now greatly reduced, though not to zero (so that $\sigma_0 > \sigma_+$), because negative charge must still escape.

3.5—Actually the excess charge, here represented by σ_+, is spread over a larger volume, between the planes $y = \pm a$, outside the stream; it includes the escaping charge referred to in § 3.4. This outside charge is equivalent, so far as regards its capacity to affect the electrostatic field within the stream, to a *smaller* amount of charge on the surface, the reduction being greater, the more distant the situation of the external volume charge density. If we ignore this extension of the excess charge, the z-component of the field outside the stream is altered approximately to the value

(12) $$4\pi\sigma_+ \,(a/r) \sin\theta - 2\pi\sigma_0\,(a/r)^2 \sin 2\theta,$$

which at the surface reduces to $4\pi\sigma \sin\theta$; it changes sign with σ, so that the force always impels the surface-charge (positive or negative) away from the stream. They will escape continuously in the $\pm z$-directions, remaining approximately between the planes $y = \pm a$. They will also have systematic motions in the x-direction, given by $c\mathbf{E}_\wedge\mathbf{H}/H^2$. As they move away from the plane $z = 0$, the corresponding values of r and θ (at least for $y > 0$) will increase. Since the second item in (12), which is negative for $y > 0$, decreases more rapidly than the first, (12) will become positive beyond a certain distance, and the electrons will gradually be retarded until their speed tends, from above, to the same limit as the increasing speed of the ions. At a great distance, they will both travel on together towards infinity with a finite speed. It may be noted that the neglect of the first-escaped electrons, in deriving (12), will be of increasing importance for large values of $\pm z$. This is evident from the fact that the potential of the field calculated from the surface distribution $\sigma_+ - \sigma_0 \cos\theta$ would contain the term $-4\pi\sigma_+ a \log r$, which is infinite at infinity. This is really balanced by an infinite term of like logarithmic order, due to the electrons that escaped first.

3.6 -The value of σ_+/σ_0 can be approximately deduced from calculations of the rate of escape of ions at the surface. The mean z-velocity of the charges in the first stage of their motion (up to a given small time, t_0 say) will be proportional to the acceleration in the z-direction, namely, F_z/m, where m denotes the mass (m_i or m_e), and F_z the z-component of the force at the surface given by (12), or $4\pi\sigma \sin\theta$. The rate at which the particles leave an element dS of the surface at a point (a, θ) is therefore proportional to $(1/m)\, \sigma^2 \sin\theta\, dS$. The electron-layer ranges from $\theta = -\theta_0$ to $\theta = \theta_0$, where $\cos\theta_0 = \sigma_+/\sigma_0$. Hence the equation of equality of rate of escape of electrons and ions is

(13) $$\frac{1}{m_e}\int_0^{\theta_0} \sigma^2 \sin\theta\, dS = \frac{1}{m_i}\int_{\theta_0}^{\pi} \sigma^2 \sin\theta\, dS;$$

on integration this gives

(14) $$\frac{\sigma_0^3 - \sigma_+^3 - 3\,\sigma_0\sigma_+\,(\sigma_0 - \sigma_+)}{\sigma_0 + \sigma_+^3 + 3\,\sigma_0\sigma_+\,(\sigma_0 + \sigma_+)} = \frac{m_e}{m_i}$$

On putting $\sigma_+ = \sigma_0\,(1 - \epsilon)$, we get

(15) $$\epsilon = 2\left(\frac{m_e}{m_i}\right)^{\frac{1}{3}} \Big/ \left\{1 + \left(\frac{m_e}{m_i}\right)^{\frac{1}{3}}\right\}$$

and since m_e/m_i is small, it follows that

(16) $$\epsilon \equiv \frac{\sigma_0 - \sigma_+}{\sigma_0} = 2\left\{\frac{m_e}{m_i}\right\}^{\frac{1}{3}}$$

approximately. For hydrogen-ions this gives $\sigma_+/\sigma_0 = 0.86$, and θ_0 about 30°, while the maximum negative value of σ is about $(1/10)\,\sigma_0$ or one-twentieth the maximum positive value of σ. For heavier ions, σ_+/σ_0 will be still nearer to unity.

These estimates clearly depend only on m_e/m_i, and not on the size or density of the stream, or the intensity of the magnetic field.

It is evident that the electrons will still be accelerated much more than the ions, and will travel away with much higher velocities. Hence their volume-density after escape will be less than that of the ions, and as equal numbers of ions and electrons must cross any plane $z = z_1$, the number of ions between two planes must exceed that of the electrons. Hence, as indicated in § 3.4, the σ_i-charge is really spread over a great range of z, and not merely over the surface of the stream. This will reduce the actual excess positive surface-charge on the stream. The electrons will be retarded after attaining a certain value of $\pm z$, while the ions are being continually accelerated, until they and the electrons finally reach a stage where they travel on side by side with equal speeds.

3.61.—The volume-density of the ions decreases as z increases, since they are being accelerated. The converse reason indicates that for the electrons the volume-density increases with z. The limiting volume-densities will be such that the number of ions and electrons between two z-planes (for large values of z) are equal. The electrons will be more concentrated than the ions because the breadth of the stream from which they escape is less than for the ions, and moreover the latter will execute larger spirals than the electrons, due to their thermal motions and their original motion $u\mathbf{x}$.

At large values of z the *mean* x-velocity of the ions and electrons will be determined by the y-component of electric force there existing. This again will depend on the total charge of either sign per unit area in the xz-plane, the part of the field due to the surface charges over the cylinder probably being negligible in comparison with the local field.

3.7.—We have not attempted to make a definite calculation of the motion and density of the escaping charges, but have obtained an estimate of the limiting z-velocity as follows: The potential over a large surface beyond the outermost escaped charges must be zero. Between $y = \pm a$, at points between the outermost electrons that escape first, and the ions and electrons and ions that follow after, the potential must be negative. At the surface of the stream, in the positively charged region, the potential is positive. The ions and electrons that leave the surface of the cylinder do so with no initial z-component of velocity, and ultimately (apart from the first set of electrons balanced by the σ_+ distribution) they attain the same z-velocity w_∞. In each case the kinetic energy which they gain is equal to the change in their potential energy between the two states. The change of kinetic energy for the ions is approximately $\frac{1}{2} m_i (w_\infty^2 + u_\infty^2 + v_\infty^2 - u^2)$, or approximately $\frac{1}{2} m_i w_\infty^2$, since the change in the x- and y-components will be slight. Similarly for the electrons it is approximately $\frac{1}{2} m_e w_\infty^2$.

Let ϕ_-, ϕ_+ be the mean potentials over the respectively negative and positive regions of surface-charge on the cylinder, and let ϕ_∞' be the potential at a great distance in the $\pm z$-direction (between $y = \pm a$), but on the nearer side of the set of electrons that escape first. Let ϕ_∞ be the potential far beyond this set of electrons. Then if ϕ_∞ is taken to be zero, ϕ_∞' will be negative, while ϕ_-, ϕ_+ will be respectively negative and positive. Since approximately

$$\tfrac{1}{2} m_i w_\infty^2 = e(\phi_+ - \phi_\infty'), \quad \tfrac{1}{2} m_e w_\infty^2 = -e(\phi_- - \phi_\infty')$$

it follows that

$$\frac{-\phi_- + \phi_\infty'}{\phi_+ - \phi_\infty'} = \frac{m_e}{m_i} \qquad (17)$$

Consequently ϕ_- is nearly equal to, but numerically rather greater than,

ϕ_∞', and $\phi_+ - \phi_\infty'$ is therefore approximately equal to $\phi_+ - \phi_-$. The determination of this difference is uninfluenced by the σ_+ distribution, and therefore only the distribution $-\sigma_0 \cos\theta$ need be considered. On averaging over the arcs from 0 to θ_0 and θ_0 to π, this leads to the approximate value $4\sigma_0 a$ for $\phi_+ - \phi_-$. Consequently

$$\tfrac{1}{2} m_i w_\infty^2 = 4ea\sigma_0$$

or

$$w_\infty = 4\sqrt{(\sigma_0 ea/2m_i)} = 4\sqrt{(aeuH/\pi cm_i)} \tag{18}$$

It should be noted that this is independent of the density of the stream.

For a very large stream w_∞ may be considerable, even in a weak field. For example, suppose $u = 10^8$ cm/sec, $a = 10^{12}$ cm (the value estimated by Chapman for solar streams at the Earth's distance) and $H = 10^{-6}$ gauss or 0.1γ, the intensity which exists at a distance of 45 Earth-radii, or about 3×10^{10} cm, from the Earth's centre; then for hydrogen-ions, $w_\infty = 10^9$ cm/sec approximately. This calculation has no immediate application to the case of the Earth, however, because the stream-width is much greater than the distance from the Earth's centre and the field over the stream would be anything but uniform. For this reason also it is doubtful whether it is worth while to attempt to find the volume-density of the escaping sets of charges in the above problem. This density is of interest for auroral theory, and it seems likely that it will not be greater than the number density of the stream itself (N), and may be very much less. One method of calculation suggests that its order of magnitude will be that of uH/eac when this does not exceed N; with the above values of u, a, H, this is about 10^{-11}.

In a non-uniform field like that of the Earth, where there is strong convergence of the lines of force, along which the escaping charges travel, an initially small density may be much increased by this cause. But the detailed consideration of the auroral aspects of our problem is deferred to a later communication.

3.8—It seems likely that the cylindrical stream will continue to emit electrons and ions at its surface until it becomes completely dispersed, tending, in fact, to become a polarized infinite slab-stream in which, besides the forward motion, there is also a non-uniform distribution of motion in the z-direction. The energy of the z-motion of the escaping charge will be supplied at the expense of the kinetic energy of the forward motion of the stream, so that the particles in the interior will be continually (though probably slowly) retarded, whilst the rate of escape of charges from the surface will gradually diminish.

The sequence of events in the case of a collection of ions and electrons streaming transverse to the field, in the form of a cylinder of *any* cross-section, is likely to be similar in all essentials to those here described for a circular cylinder.

4—*The motion of a neutral ionized system of any form in a uniform magnetic field*

4.1—Any uniform neutral system of N ions and N electrons per unit volume, moving with a uniform velocity $\mathbf{V}$ in a uniform magnetic field $\mathbf{H}$, will become polarized by a relative displacement $\mathbf{dr}$ of the ions and electrons, such as will set up a uniform electric field $\mathbf{E} = -(1/c)\,\mathbf{V}\wedge\mathbf{H}$ almost exactly, within the system, whatever its outer form.

This field will balance the electromagnetic deflecting force on the charges, which will therefore move (almost exactly) along rectilinear paths. The displacements of the ions and electrons relative to their undisturbed paths (that is, their paths in the absence of the magnetic field) will be approximately inversely proportional to their masses, so that the relative motion will be nearly wholly due to the electrons. The polarization $\mathbf{P}$ is given by $Ne\mathbf{dr}$; it will depend on the outer form of the system, but not on the size, and in general will vary in magnitude and direction from point to point; but in three simple special cases it is uniform. These correspond to the outer forms (*a*) an infinite-plane slab, for which $P = \frac{1}{4\pi c} \mathbf{V}_\wedge \mathbf{H}$, (*b*) an infinite circular cylinder, for which $\mathbf{P} = (1/2\pi c) \mathbf{V}_\wedge \mathbf{H}$ and (*c*) a sphere, for which $P = (E/4\pi c) \mathbf{V}_\wedge \mathbf{H}$. It is here supposed that $\mathbf{dr}$ is small and that the surface-charge, which will be the component (P_n) of $\mathbf{P}$ normal to the surface, has not had time to be dispersed by its mutual repulsion.

The volume-distribution of charge will be zero in all cases, since the electric field is uniform. The surface-distribution of charge (necessary to produce this electric field $\mathbf{E}$) will be the same as that induced on a conductor of the same shape as the stream by an external field $(1/c)$ $\mathbf{V}_\wedge \mathbf{H}$.

4.2—Except in the case of the infinite-plane slab, the surface-charge is repelled from the surface, and more rapidly from the negatively charged area than from the positive, until the system has acquired, by this process, an excess positive charge, thus enlarging the positive surface area at the expense of the negative. The dispersed surface-charge is, however, continually replaced by the motion of charges (mainly of electrons) to or from the surface, so as to maintain the internal field almost exactly at the value $-(1/c) \mathbf{V}_\wedge \mathbf{H}$. The difference between $\mathbf{E}$ and this value must be such as will produce the necessary flow of the electrons to maintain the surface-charge.

4.3—The three special cases in which $\mathbf{P}$ is known suggest that, in general, the order of magnitude of $\mathbf{P}$ is $(1/2\pi c) \mathbf{V}_\wedge \mathbf{H}$, so that $\mathbf{dr}$ will be of the order of $(1/2\pi c Ne) \mathbf{V}_\wedge \mathbf{H}$. This is greatest when $\mathbf{V}$ is perpendicular to $\mathbf{H}$, and then $\mathrm{d}r$ is of the order $VH/2\pi cNe$, or approximately $VH/90N$. Thus taking $H = 0.35$, the surface-value at the Earth's equator, and $V = 10^8$, $\mathrm{d}r = 4 \times 10^5/N$, so that if $N > 0.1$, $\mathrm{d}r < 40$ km; this is very small compared with the Earth's dimensions.

4.4—If the magnetic field, though remaining uniform, changes with time, the stream will remain approximately in the steady state, provided that the rate of change of the field is not too large. The exactness with which the field E is likely to balance the electromagnetic force on the charge may be illustrated by determining the time required for the displacement $\mathrm{d}r$ to be produced, supposing that $\mathbf{E}$ falls short of the exact balancing field $-(1/c) \mathbf{V}_\wedge \mathbf{H}$ by a fraction f, necessary to produce the required change in polarization. The transverse acceleration of the electrons is then $feVH/m_e c$, where m_e denotes the electronic mass. The corresponding time δt required for the transverse displacement $\mathrm{d}r$ is given by

$$(19) \qquad (\delta t)^2 = 2\, m_e c\, \mathrm{d}r/fe\, VH = m_e/f\pi e^2\, N,$$

on substituting for $\mathrm{d}r$ from § 4.3.

Inserting numerical values, this gives $\delta t = 3.5 \times 10^{-5}/\surd(f N)$ so that if, for example, $f = 1/1000$, and $N = 0.1$, then $\delta t = 3.5 \times 10^{-3}$ sec. If this time is small compared with that during which the intensity of the field H alters in a sensible ratio, the stream will approximately take up the "equilibrium" state at every instant.

4.5—If the field, instead of being uniform, increases in intensity in the direction of motion, then it may be possible for a steady state to be set up as in § 4.4. In this case the displacement dr appropriate to each point will increase as the stream advances, and the deflecting force will likewise increase; the value f of the unbalanced fraction of this force will adjust itself so that the "equilibrium" value of dr is nearly attained at each point. This requires that the distance traversed during the interval δt in the direction of the stream, namely, $3.5 \times 10^{-5} V/\surd(fN)$, shall be small compared with that over which the intensity of the field H alters in a sensible ratio. In the Earth's field, H varies as $1/r^3$, so that a change of 10 per cent in H is attained when r changes by about 3 per cent. This distance is least near the Earth's surface. Three per cent of the Earth's radius is 2×10^7 cm or 200 km, which is certainly large compared with $3.5 \times 10^{-5} V/\surd(fN)$ cm even if (when $N = 0.1$ and $V = 10^8$ cm/sec), f is 10^{-6}. Only if N were very much less than 0.1, that is, for an extremely rare stream, would the unbalanced fraction of the deflecting force have to be other than small, in order to provide the equilibrium polarization by deflection of the electrons. But when N is very small, dr is correspondingly large, and the conception of the displacement as constituting a polarization ceases to have value. It seems probable, as will appear later, that such rare streams will have no appreciable effect upon the Earth's magnetic field.

4.6—In the preceding discussion the field has, in the main, been supposed constant as well as uniform, and the polarization has been supposed established from the outset. But if the field is slowly increased from zero, or if the stream advances into regions of gradually increasing intensity, the polarization is set up gradually. One effect of this is to retard the stream. The order of magnitude of the retardation can be calculated from the principle of conservation of energy. The polarization sets up an electric field of intensity $(1/c)\ \mathbf{V}_\wedge \mathbf{H}$, with which is associated electrostatic energy of amount $E^2/8\pi$ or $V^2H^2/8\pi c^2$ per unit volume; it also increases the intensity of the magnetic field, for, as has been shown in § 2.51, $H^2 = H_0^2 + V^2H_0^2/c^2$, approximately, inside the stream. The increase in magnetic energy $(H^2 - H_0^2)/8\pi$ is therefore the same as the electrostatic energy. The total increase in the field-energy, $V^2H^2/4\pi c^2$ per unit volume, must be drawn from the kinetic energy of the stream, which is therefore reduced in velocity from (say) V' to V, where

(20) $$\rho\ (V'^2 - V^2) = V^2\ H^2/4\pi c^2,$$

ρ being the mass-density of the stream.

Hence

(21) $$(V/V')^2 = \lambda/(1+\lambda)$$

where

(22) $$\lambda = 4\pi\rho c^2/H^2$$

This formula for the retardation is similar to one already given by Chapman,[19] which was obtained by considering the reaction of the magnetic field upon the transverse currents that set up the increasing polarization.

The following table indicates how the retardation varies with λ. The corresponding values of ρ, taking $H = 0.35$ (the surface-value at the Earth's equator), are also given. For smaller values of H, ρ for any given value of λ is reduced in proportion to H^2.

λ	$\frac{1}{100}$	$\frac{1}{10}$	$\frac{1}{5}$	1	2	10	100
ρ	1.07×10^{-25}	1.07×10^{-24}	2.13×10^{-24}	1.07×10^{-32}	2.13×10^{-23}	1.07×10^{-22}	1.07×10^{-21}
V/V'	0.10	0.30	0.41	0.71	0.82	0.95	1.00

It may be noted that a stream containing one hydrogen atomic ion (and one electron) per cc would have a density of 2.1×10^{-24} gm/cc. Such a stream would be considerably retarded in traveling into a field of intensity 0.35. The retardation becomes small, however, for streams of density corresponding to $50H^+$ ions (or about 1 Ca^+ ion) per cc.

4.7—So far it has been supposed that the velocity of the stream is uniform throughout the stream, so that in general a steady state could approximately be set up. But if the velocity of the stream is not uniform, such a steady state cannot be set up, even in a uniform magnetic field, unless or until the velocity of the stream can be rendered uniform. The reason for this is that the deflecting force acting on the charges, namely, $(1/c)\ \mathbf{V}_\wedge \mathbf{H}$, will not be derived from a potential, so that no distribution of electric charges could produce the balancing electrostatic field necessary for a steady state. This case arises, for example, in the motion of an infinite cylindrical stream in a uniform magnetic field when the velocity of the stream varies in the direction of the field. Provided the density is not too low, electric currents will then flow across the cylinder, which will be acted on by the magnetic field. The main body of the stream will thus be accelerated in some parts and retarded in others, in a manner tending to make the velocity uniform throughout the stream.

5—*The internal steady motion of a neutral ionized stream in a non-uniform magnetic field*

5.1—The next problem to be discussed is the one already considered by Chapman,[19] namely, the internal *steady* motion of a neutral ionized stream in a *non-uniform* magnetic field. The region over which the latter extends (that is, over which the charges in the stream are deflected appreciably by the field) is supposed small compared with the lateral dimensions of the stream. The latter is supposed infinite along the direction of motion, the charges moving almost exactly along parallel straight lines, except in the region of the field.

The discussion of the problem at this stage is in accordance with the sequence of argument in this paper, though, as will appear, the important magnetic effects of the stream directed towards the Earth are phenomena of the initial stages, and not of the possible final steady state.

The difficulty of determining such a steady state is almost insuperable, the problem being, in fact, far more complicated than that of the *approach* of the stream towards the magnetic system, and here we shall only make some general remarks as to the nature of the solution.

It is obvious that if the stream is so rare that at any instant comparatively few charges (ions and electrons) are within the whole region of the field, their motions will be practically independent of one another, and approximately the same as those of a solitary charge, such as is considered in Störmer's researches. This is one extreme case; the other is that in which the electromagnetic deflection by the field is confined within narrow limits by the internal electrostatic field set up by the deflection itself, and also by the interaction of the charges between themselves. This second extreme case does not require a large density, $N = 0.1$ being amply sufficient for a stream moving with a velocity of 10^8 cm/sec in a field such as the Earth's (§ 4.5).

5.2—It might be supposed that in such cases the stream can become polarized at each point—a hypothesis that suggests itself by analogy with the simpler cases considered in §§ 2-4 and which, in fact, was assumed by Chapman in his investigation; but on closer examination this hypothesis proves to be untenable in the general case.

The assumption that the stream is polarized implies that the two sets of opposite charges in any small volume-element of the stream move approximately together. On account of the high mobility of the charges, this requires that the electric field **E** should balance almost exactly the magnetic deflecting force acting on the charges, which may be approximately taken as $(1/c)\mathbf{V}_\wedge\mathbf{H}$, ($V$ being the mean velocity of the charges in the element), unless this be vanishingly small. The electric field will therefore be independent of the density (N) of the stream, except in so far as this may affect the variation of **V**. Since the polarization **P** must produce this field, **P** will likewise be nearly independent of N.

By the principle of conservation of energy, the electromagnetic energy produced in setting up the polarization of the stream must be supplied by the mechanical or kinetic energy of the stream itself. The stream would therefore be retarded, the retardation being greatest in the strongest regions of the magnetic field. But since the polarization is independent of N, it is clear that a sufficiently dense stream would hardly be retarded at all. For a stream moving with velocity 10^8 cm/sec in the Earth's magnetic field, it would be sufficient for N to exceed a density of 10^{-22} gm/cc (corresponding to about 50 H or 1 Ca-atom per cc), to judge from § 4.6. Similar reasoning shows that the bending of such streams by the magnetic field would be exceedingly small, since the currents, due to the polarization, would be independent of N. Thus in a dense stream (such as has just been considered) the charges would move approximately along their undisturbed paths; and this leads at once to an inconsistency unless the interior of the stream can be shielded from the magnetic field. For the electric field, as we have seen, must be approximately equal to $(1/c)\ \mathbf{V}_\wedge\mathbf{H}$, and in the particular case of a steady state it must satisfy the condition **cur E** $= 0$. Since **V** is approximately uniform, and since div $\mathbf{H} = 0$, this reduces very nearly to the condition

$$(\mathbf{V}.\ \mathbf{grad})\ \mathbf{H} = 0,$$

or

$$\delta\mathbf{H}/\delta s = 0,$$

where $\delta/\delta s$ denotes differentiation in the direction of **V**. This requires that **H** is uniform along any stream-line. But **H** tends to zero at large distances, hence it must vanish at all points in the stream; but this is contrary to the initial hypothesis that H exists and polarizes the stream. Thus we must abandon the hypothesis for the most likely case of all (that of a dense stream) and so also in general—always, in fact, except for very special cases, such as those considered in §§ 2-4, where $\delta H/\delta s = 0$, because H is uniform throughout the stream.

5.3—The only alternative is to suppose that a system of steady currents flows within the stream, their distribution and intensity depending mainly on the external field and the velocity and density of the stream. They may or may not shield the interior from the external magnetic system.

In any actual stream, however, the mutual interaction of the charges would have to be taken into account (except for the extreme case already referred to). If the stream be sufficiently dense, the conditions would be much the same as those of a compressible conducting medium. This would be the next simplest case to consider, since in the interior of any isotropic conductor the volume-density of charge is vanishingly small. The electrostatic field within such a stream would be negligible, and the current density at any point would be proportional to, and in the direction of, the electromotive force $\mathbf{V}_\wedge \mathbf{H}$ (in e.m.u.) produced by the motion of the stream in the magnetic field. It is supposed that the positive and negative charges move with nearly (though not quite) the same velocity V, which requires the density to exceed some very small lower limit.

The conductivity of the stream is independent of its density, so that the current is proportional to $\mathbf{V}_\wedge \mathbf{H}$, the same for all densities above the lower limit mentioned (except so far as the changes in V depend on the density). The reaction of the field on the current would bend the stream except in those regions where $\mathbf{V}$ and $\mathbf{H}$ are in the same direction. The force tending to produce bending is nearly independent of the density, and hence the bending itself will be nearly inversely proportional to the density.

It can be verified that the tendency would be for the stream to converge towards the poles of the magnetic system, and that the flow of currents would be such as to diminish the magnetic field in the regions where the stream moves towards the magnetic system, possibly diminishing the field over the magnetic system also. In the further regions where the stream moves away from the system, the magnetic field would be increased somewhat, decreasing steadily to zero at infinite distance.

In the case of streams whose conductivity is not isotropic (for example, in a stream whose conductivity in the direction normal to the magnetic field is reduced because of spiral motion of the charges between collisions), it would be possible for a volume-distribution of charge to be set up within the interior. The problem in this case is thereby complicated still further; but fortunately in the theory here being developed the importance of the problem of the steady state is not sufficient to require a detailed investigation of it at present.

(*To be continued*)