The limits of Earth orbital calculations for geological time-scale use

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The orbital motion of the planets in the Solar System is chaotic. As a result, initially close orbits diverge exponentially with a characteristic Lyapunov time of 5 Ma. This sensitivity to initial conditions will limit the possibility of obtaining an accurate solution for the orbital and precessional motion of the Earth over more than 35–50 Ma. The principal sources of uncertainty in the model are reviewed here. It appears that at present the largest source of error could reside in the lack of knowledge of the value of the precession due to the oblateness (J_2) of the Sun. Nevertheless, for the calibration of geological time-scale, this limitation can be overcome to some extent if one considers in the geological data the signature of the outer planets' secular orbital motion which is predictable on a much longer time-scale. Moreover, it should be possible to observe in the geological records the trace of transition from the $(s_4 - s_3) - 2(g_4 - g_3)$ secular resonance to the $(s_4 - s_3) - (g_4 - g_3)$ resonance. The detection and dating of these passages should induce extremely high constraints on the dynamical models for the orbital evolution of the Solar System.

**Keywords: precession; obliquity; orbital evolution; chaos;
Solar System; Milankovitch cycles; palaeoclimates**

1. Introduction

The insolation at a given point on Earth depends on the position of the Earth in space, and on the orientation of the Earth relative to the Sun. In order to make precise computations of the past climatic evolution of the Earth, one thus needs first to have an accurate solution for the orbital evolution of the Earth, and then to compute the evolution of the orientation of the axis of the Earth.

The orbital computation is a difficult task since the Earth's motion is perturbed by all the other planets of the Solar System. The first approximate solution of this problem was given by Le Verrier (1856), and was used by Milankovitch for his studies on the astronomical origin of the ice ages. Le Verrier's solution consisted of linearized equations for the mean evolution of the orbits of the planets. It was completed later on by Hill (1897), who recognized that higher-order terms, coming from the mutual interactions of Jupiter and Saturn, can significantly change the solution of Le Verrier. The solution of Brouwer & Van Woerkom (1950) is essentially that of Le Verrier, with the addition of the terms computed by Hill. It provides a solution which represents in a very reasonable way the evolution of the orbit of the Earth over a few hundred thousand years, and was used for insolation computations by Sharav & Boudnikova (1967a, b) with updated values of the parameters. The next major improvement was given by Bretagnon (1974), who computed terms of second order and degree 3 in the

secular (mean) equations. This solution was then used by Berger (1976, 1978) for the computation of the precession and insolation quantities for the Earth following Sharav & Boudnikova (1967*a, b*). All of these solutions assumed implicitly that the motion of the Solar System is regular, and that the solution could thus be obtained as quasi-periodic series, using perturbation theory.

In Laskar (1984, 1985, 1986), I computed in an extensive way the secular equations giving the mean motion of the whole Solar System, including all terms up to order 2 with respect to the masses, and up to degree 5 in the eccentricity and inclination of the planets (thus also including all the terms of Hill). It was clear from these computations that the assumed practical convergence of the perturbative series was vain, and strong evidence of divergence becomes apparent in the system of the inner planets (Laskar 1984).

This difficulty was overcome by a numerical integration of the secular (i.e. averaged) equations, which could be performed in a very efficient way, using a stepsize of 500 years. The outcome of these computations was to provide a much more accurate solution for the orbital evolution of the Solar System (Laskar 1986, 1988), which also included a full solution of obliquity and precession usable for insolation computation. But at the same time, I demonstrated the chaotic behaviour of the orbits of the planets of the Solar System, and more specifically of the inner planets, thus destroying the hope of obtaining an astronomical solution to use to develop a palaeoclimate time-scale for the Earth over several hundreds of millions of years (Laskar 1989).

Later on, and shortly after the publication of the direct numerical integration over 3 Ma of Quinn *et al.* (1991), we published an improved solution for the precession and obliquity solution, aimed to help palaeoclimate computation over 20 Ma (Laskar *et al.* 1993*a*). This solution was made widely available through the Internet (requests for these files should be addressed to laskar@bdl.fr), together with a set of routines allowing changes to the model of precession. This solution was limited to 20 Ma, that is the time span over which we assumed that the exponential divergence due to chaotic behaviour was not sensitive. Due to the improvement of the geological techniques, several groups are now studying sedimentary cores extending over more than 20 Ma, and there is a demand for astronomical solutions extending further in time. This requires a closer look at the real precision of the solutions, and the aim of the present paper is to clarify this point, by setting more precise bounds on the limits of Earth orbital calculations for geological time-scale use.

2. Chaotic motion of the Solar System

The main result from the long-term integration of the secular Solar System equations was the discovery that the full Solar System, and especially the inner Solar System is chaotic (Laskar 1989) with a Lyapunov time of *ca.* 5 Ma. This means that the distance of two planetary solutions, starting in the phase space with a distance $d(0) = d_0$, evolves approximately as†

$$d(T) \approx d_0 \mathrm{e}^{T/5}, \tag{2.1}$$

† The distance here corresponds to the variables used in the integration of the problem, which are either similar to an eccentricity, or to an inclination. In both cases, if the inclinations are expressed in radians, they are relative distances in physical space, but for an easier comparison of the results for the inclinations, we prefer to use degrees in the figures instead of radians.

or, in a more straightforward way, and which is even closer to the true value,

$$d(T) \approx d_0 10^{T/10}, \tag{2.2}$$

where T is expressed in millions of years. Following this formula, an initial error of 10^{-10} leads to an indeterminacy of $\approx 10^{-9}$ after 10 Ma, but reaches the order of 1 after 100 Ma. From these results, it was clear that the planetary solutions can be very accurate for 10–20 Ma, and probably irrelevant for precise predictions after 100 Ma.

Nevertheless, the Lyapunov time of 5 Ma, which is given here, is a global constant for the whole Solar System. This is valid, as due to coupling, all solutions will undergo the effects of the chaotic component; but as this coupling is small, the effect on some of the planets could also be small over a limited time span.

For the practical purpose of calibrating the astronomical time-scale for the terrestrial sediments, it is thus necessary to look more closely at the specific solution of the Earth's orbital elements. This is done in the present work through several numerical experiments.

(a) Experiment 1

In the first experiment, I changed the initial argument of perihelion of the Earth by an amount between 10^{-2} and 10^{-9} rad. After integrating the orbits over 100 Ma, the eccentricity and the inclination of the Earth are compared directly in figures 1 and 2. In figure 1, the propagation of the error in eccentricity is plotted versus time, while in figure 2, the error in inclination is given in degrees.

In all of these plots, it is clear that after a time span during which the difference of the solutions is barely seen, there is a brutal increase in the error, reaching the total amplitude of the variation of these elliptical elements. This change, which results from the dominance of the exponential chaotic component, results in a complete loss of the phase relationship at the corresponding angles of precession of node and perihelion, and thus strictly limits the use of the astronomical solution as a time-scale for geological records.

From figures 1 and 2, one gets the following simple approximate relation between the time of validity of the orbital solution T_V and the absolute error in the angles determining the initial position of the orbit of the Earth in space δ_ϖ,

$$T_V \approx -\log_{10} \delta_\varpi \times 10 \text{ Ma}, \tag{2.3}$$

which is actually equivalent to equation 2.2, the validity time T_V being obtained for an error $d(T_V) = 1$.

(b) Experiment 2

In this second experiment, the same change of 10^{-5} rad is made successively for the perihelions of all of the planets. This induces an error in the orbit of the Earth due to the coupling of the solutions. The errors in eccentricity and inclination are plotted in figures 3 and 4, respectively. From these plots, it appears that some errors in the perihelion of Mercury, Venus, Mars, Jupiter and Saturn have the same effect as an equivalent initial error in the perihelion of the Earth. So the same relation (2.3) will apply.

For Uranus and Neptune, the induced error is not so large, and we can see an offset of 30 Ma with respect to the previous relation. The time of validity of the orbit will thus be something more like

$$T_{\mathrm{V}} \approx -\log_{10}(10^{-3}\delta_{\varpi}) \times 10 \mathrm{\ Ma}, \tag{2.4}$$

which means that for Uranus and Neptune we can accept an error 1000 times larger than that for the other planets.

(c) *Experiment 3*

Here we start with an offset in the orbit of one of the planets, and examine the resulting effect for the solution of all the other planets. This is done successively for the Earth (figure 5), Jupiter (figure 6), and Neptune (figure 7). For brevity, only the results for eccentricity are plotted, but the results for inclination are very similar.

It is quite clear that all the inner planets have the same chaotic behaviour, while all of the solutions for the outer planets behave much more regularly. There are still some chaotic effects, especially in figure 5, but the resulting error over 100 Ma is still very small. With an initial error for an outer planet (figures 6 and 7), the errors in the outer planets grow regularly and do not show exponential trends over 100 Ma.

(d) *Conclusions*

From these computations, it appears clearly that the solution for the Earth follows strictly the exponential relations given in (2.1) and (2.2), and that the time of validity for the orbital solution will be given by the relation (2.3). These numerical experiments will now be used to estimate the propagated error due to the uncertainty in the initial conditions and parameters of the model.

3. Constants of the planetary solution

If one sets aside the problems due to the model, the accuracy of the orbital solution depends on the planetary masses and on the precision of the initial positions and the determination of velocities.

(a) *Planetary masses*

The precision with which masses are known has been greatly improved by the Voyager Spacecraft missions, and the latest values for the planetary masses are given in table 1. The uncertainties are obtained from the latest adjustment of the JPL ephemeris DE405 (Standish 1998).

(b) *Planetary positions*

The uncertainty of the observations for the positions of the planets should be better than $0.1'' \approx 0.5 \times 10^{-6}$ rad, and this will not be a limiting factor for obtaining a solution over an extended time span.

If we assume in general a precision of 10^{-6} for the planetary masses and positions, using the relation (2.3), one sees that the maximum validity time for the orbital solution will be

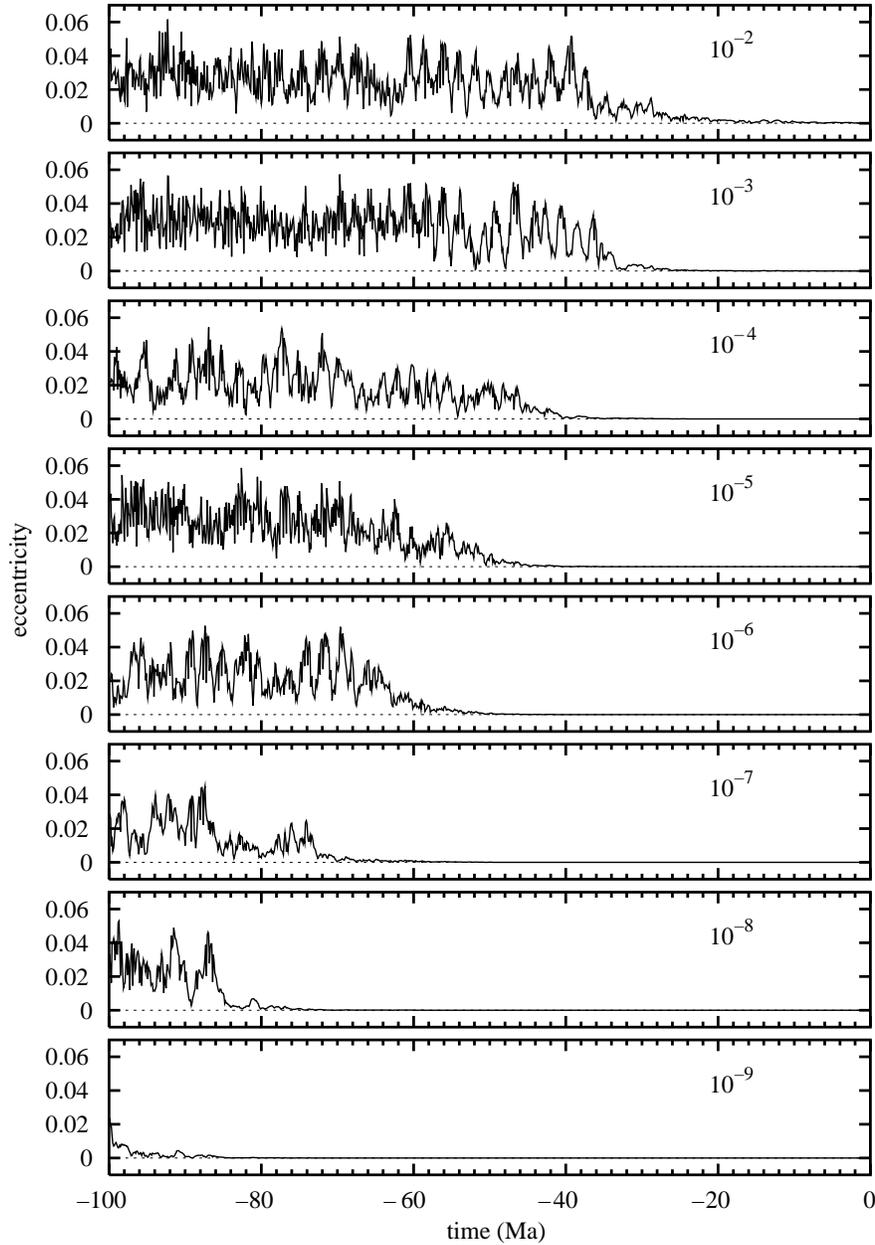$$T_{\mathrm{V}} \approx 60 \mathrm{\ Ma}. \tag{3.1}$$

Figure 1. Error in the eccentricity of the Earth resulting from an initial change of 10^{-n} rad in the perihelion of the Earth at the origin. After about $n \times 10$ Ma, the exponential divergence of the orbits dominates, and the solutions are no longer valid. Error in eccentricity is plotted versus time (in Ma).

This is somewhat optimistic, as the previous section shows that this precision of 10^{-6} needs to be achieved for all the planets from Mercury to Saturn to reach this validity time for the solution. If this were all, the situation would be quite encouraging. A
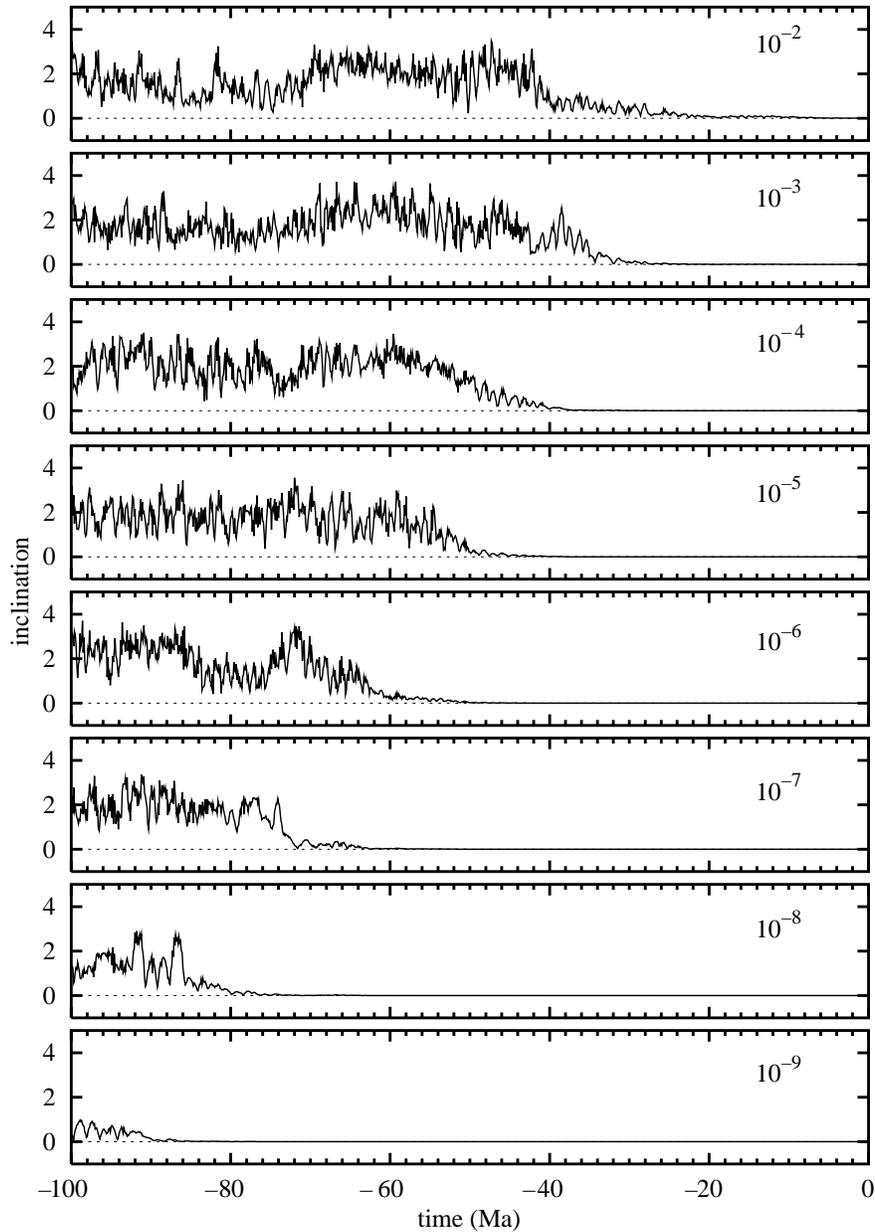
Figure 2. Error in the inclination of the Earth resulting from an initial change of 10^{-n} rad in the perihelion of the Earth at the origin. After about $n \times 10$ Ma, the exponential divergence of the orbits dominates, and the solutions are no longer valid. Error in inclination (in degrees) is plotted versus time (in Ma).

solution valid over 100 Ma would still be out of reach, but from this naive analysis, a solution valid over 50 Ma would appear to be consistent with the present accuracy of the determination of the planetary initial conditions and mass values.
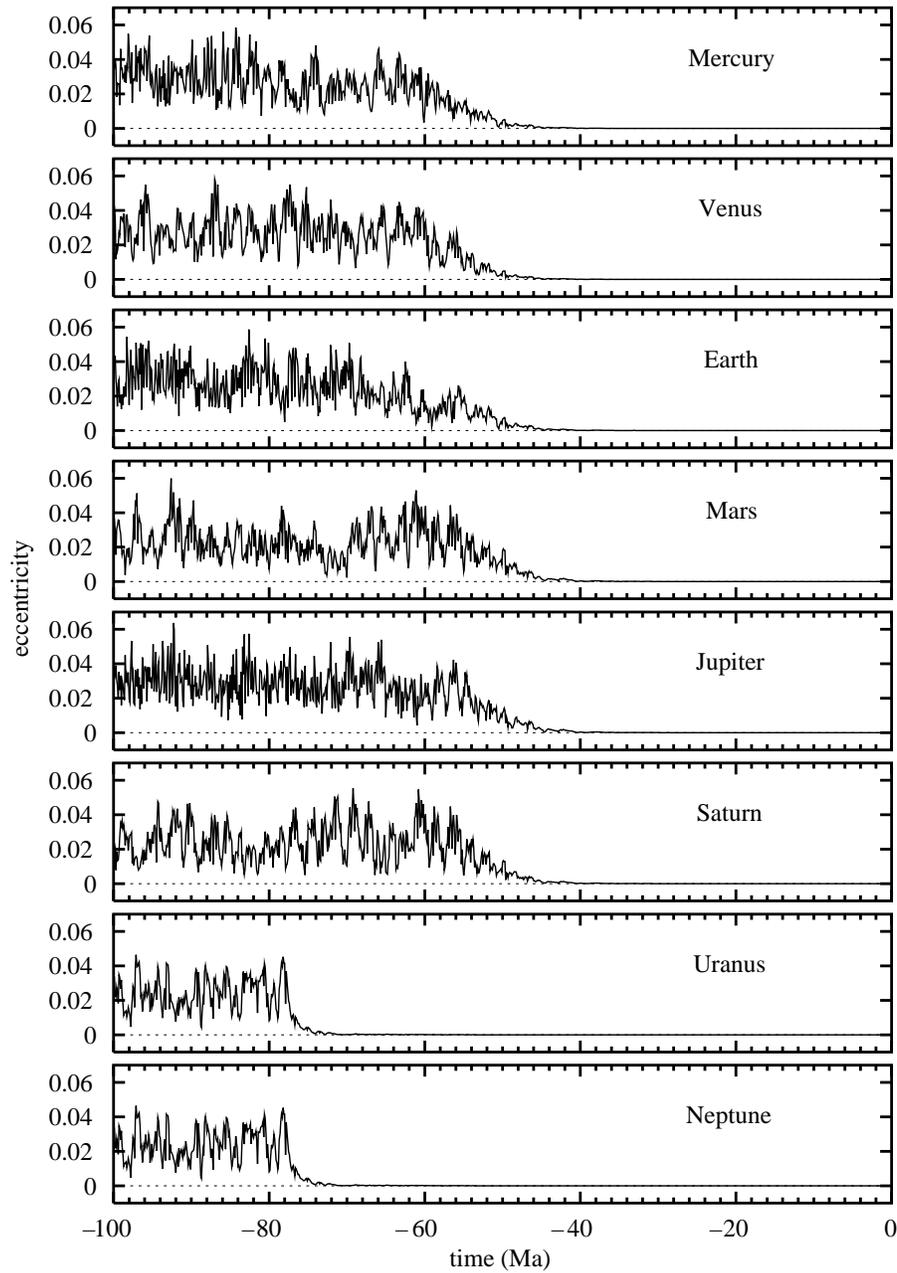
Figure 3. Error in the eccentricity of the Earth resulting from an initial change of 10^{-5} rad in the perihelion of the various planets at the origin. It is clear that an error in the position of Mercury, Venus, Mars, Jupiter, Saturn, has the same impact as the same error in the perihelion of the Earth. For Uranus and Neptune, due to the smaller coupling, the error becomes important only 30 Ma later. Error in eccentricity is plotted versus time (in Ma).
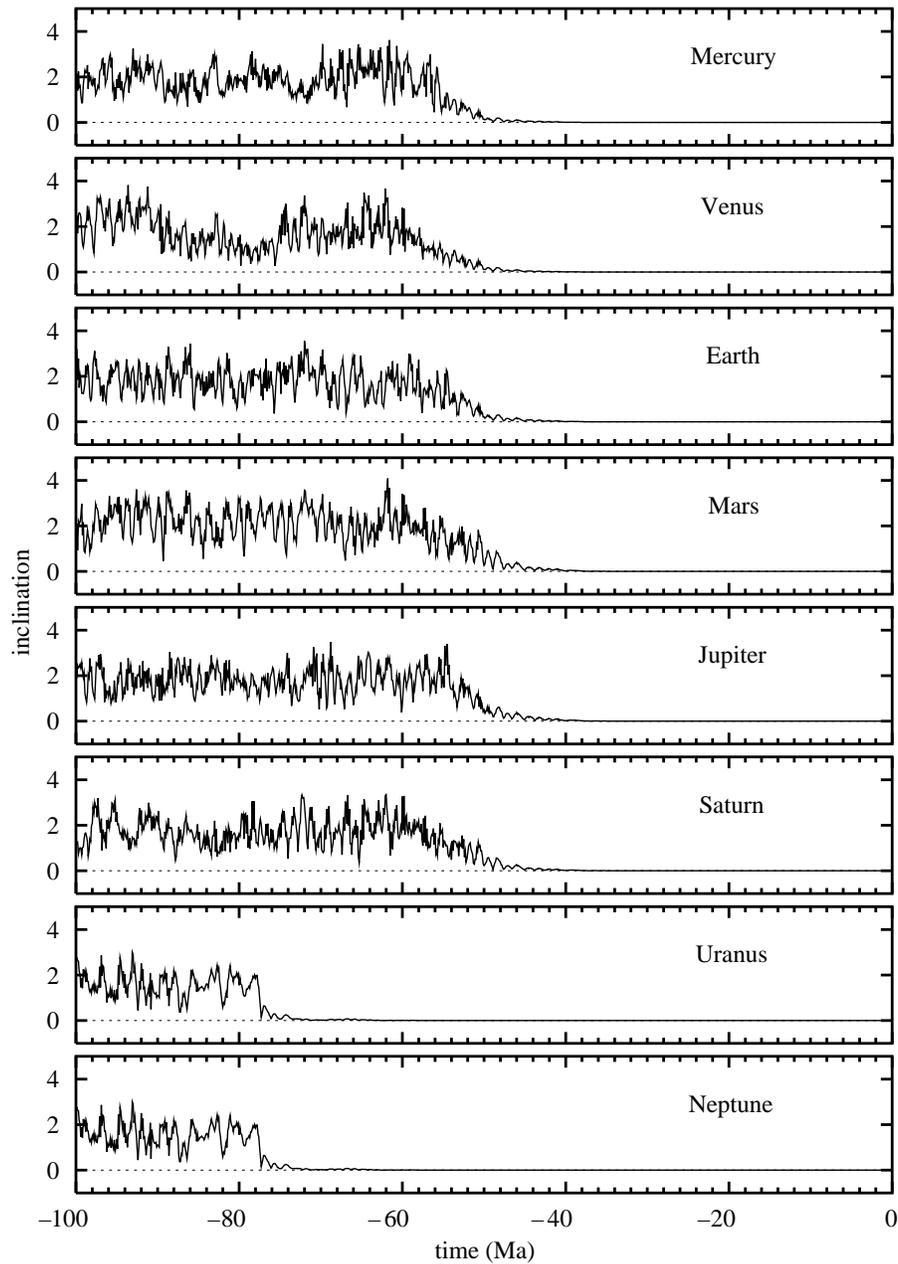
Figure 4. Error in the inclination of the Earth resulting from an initial change of 10^{-5} rad in the perihelion of the various planets at the origin. It is clear that an error in the position of Mercury, Venus, Mars, Jupiter, Saturn, has the same impact as the same error in the perihelion of the Earth. For Uranus and Neptune, due to the smaller coupling, the error becomes important only 30 Ma later. Error in inclination (in degrees) is plotted versus time (in Ma).
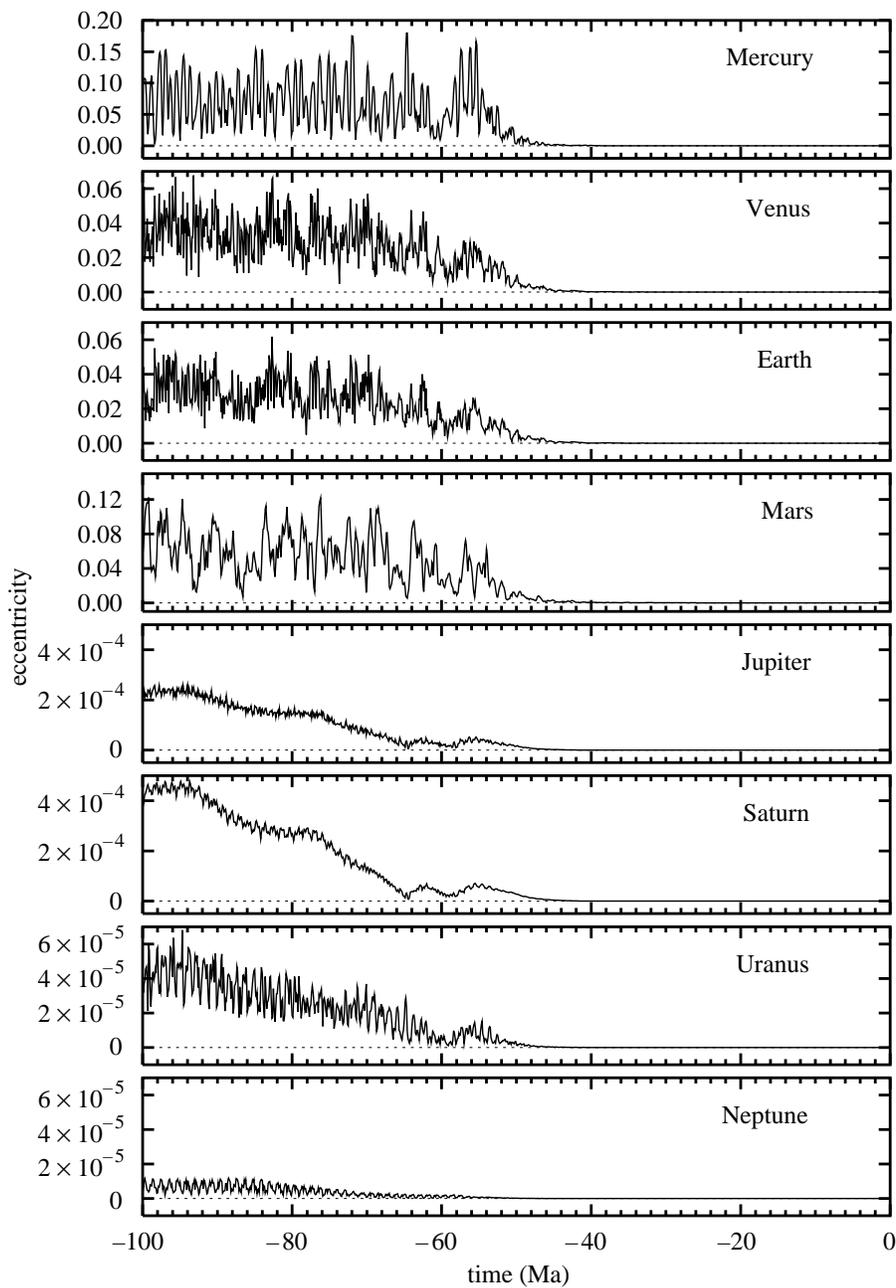
Figure 5. Error in the eccentricity of the various planets resulting from an initial change of 10^{-5} rad in the perihelion of the Earth at the origin. One can see that the effect of the chaotic behaviour is the same for all the terrestrial planets (Mercury, Venus, Earth, Mars), but due to the small coupling, it is much smaller for the outer planets, for which the effect is negligible (notice the change of scale) over 100 Ma. Error in eccentricity is plotted versus time (in Ma).
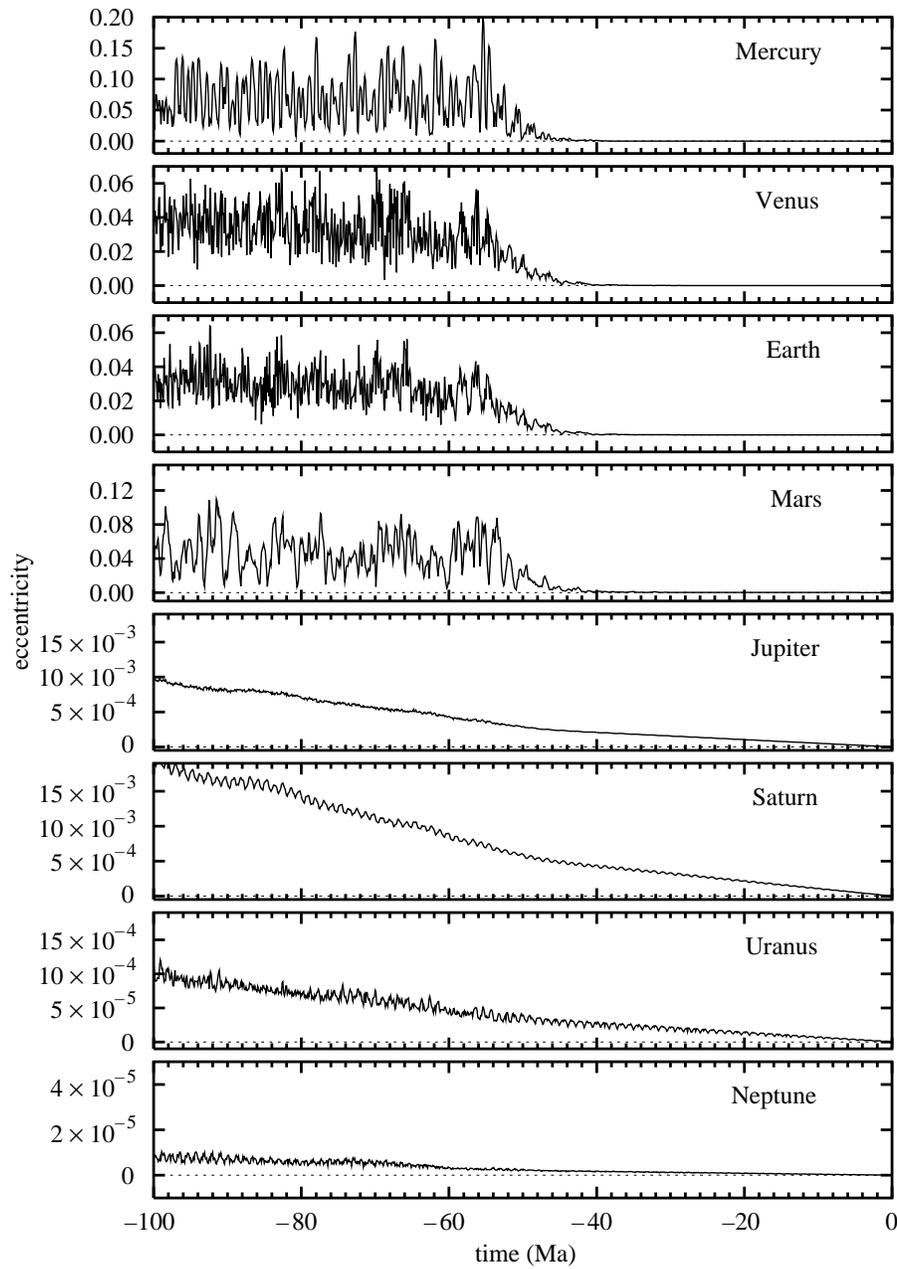
Figure 6. Error in the eccentricity of the various planets resulting from an initial change of 10^{-5} rad in the perihelion of Jupiter at the origin. One can see that the effect of the chaotic behaviour is large for all the terrestrial planets (Mercury, Venus, Earth, Mars). There is an increase of the error for the outer planets, but it follows a very regular rate, which is what we could expect in a regular problem. For the outer planets, the effect is still very small over 100 Ma. Error in eccentricity is plotted versus time (in Ma).
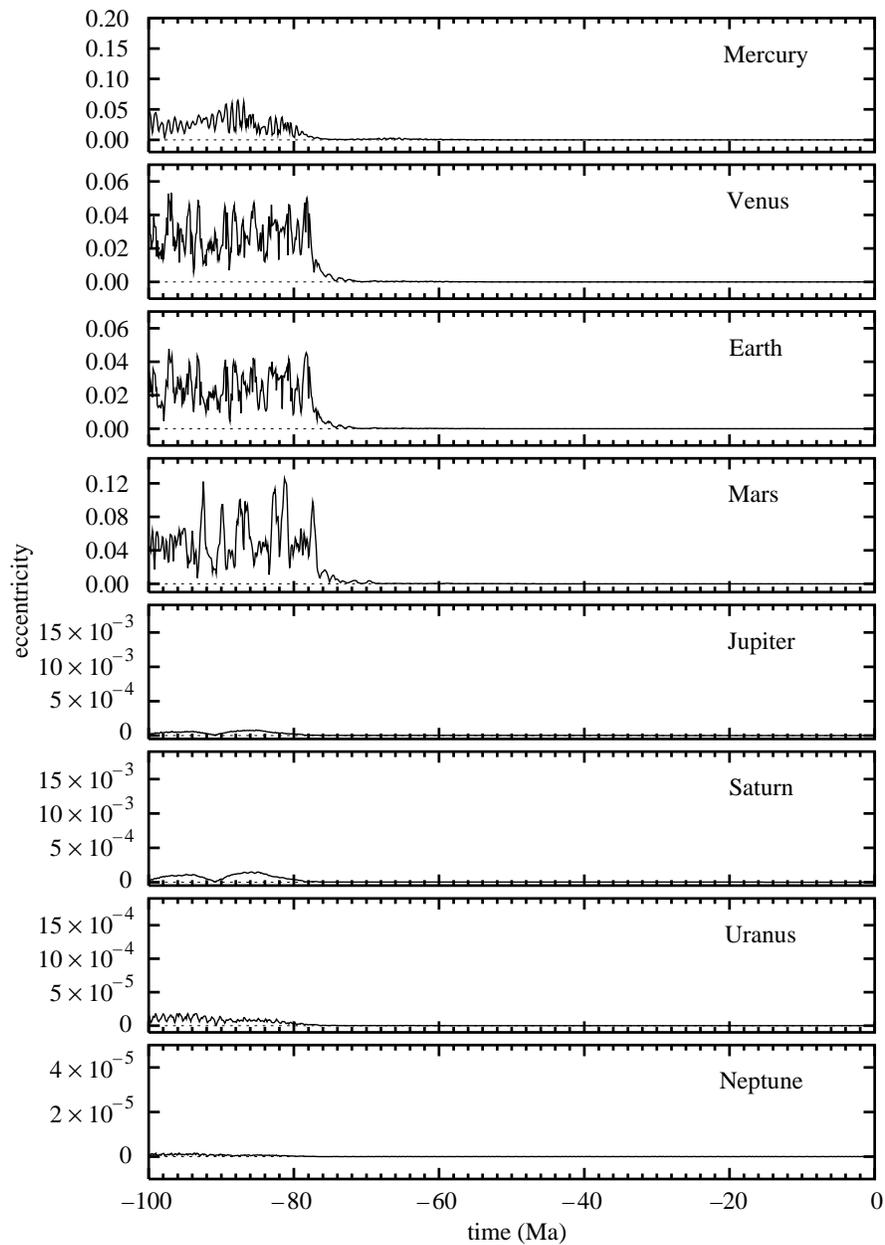
Figure 7. Error in the eccentricity of the various planets resulting from an initial change of 10^{-5} rad in the perihelion of Neptune at the origin. The effect of the chaotic behaviour is still present for all the terrestrial planets (Mercury, Venus, Earth, Mars), but of lower amplitude, as it manifests itself only after *ca.* 75 Ma. The error for the outer planets is also much smaller, and follows a very regular rate, which is what we could expect in a regular problem. For the outer planets, the effect is extremely small over 100 Ma. Error in eccentricity is plotted versus time (in Ma).

Table 1. *Value of the ratio $M_\odot/m$ for the planets of the Solar System given by IAU 1976,*
IERS 1992, DE405, and estimated relative precision $\Delta m/m$

	IAU 1976	IERS 1992	DE405	Δm	$\Delta m/m$ $\times 10^6$
Mercury	6 023 600	6 023 600	6 023 600	250	40
Venus	408 523.5	408 523.71	408 523.71	0.06	0.15
Earth+Moon	328 900.5	328 900.56	328 900.5614	0.02	0.06
Mars	3 098 710	3 098 708	3 098 708	9	3
Jupiter	1 047.355	1 047.3486	1 047.3486	0.0008	1
Saturn	3 498.5	3 497.90	3 497.898	0.018	5
Uranus	22 869	22 902.94	22 902.98	0.03	1.5
Neptune	19 314	19 412.24	19 412.24	0.04	2
Pluto	3 000 000	135 000 000	135 200 000	7 000 000	52 000

In fact, the situation is not so good, as the simple following analysis will show.

4. Precision of the secular frequencies

An error in the initial values of the planetary positions and masses also induces an error in the secular frequencies, or in an equivalent manner, in the precession rates,

$$\dot{\varpi}_j = \frac{\mathrm{d}\varpi_j}{\mathrm{d}t}, \qquad \dot{\Omega}_j = \frac{\mathrm{d}\Omega_j}{\mathrm{d}t}, \tag{4.1}$$

which we can also assume to be of the order of 10^{-6}. Let us look at the effect of an error of δ_ν in one of the precession frequencies, or angular velocities.

The error in the angle will increases as $\delta_\nu \times T$ at first, but then the exponential divergence due to the chaotic behaviour will also be important. The error will thus grow approximately as

$$d(T) = \delta_\nu \times T_1 \mathrm{e}^{(T-T_1)/5}. \tag{4.2}$$

The value of T_1 which gives the maximum value for this error is $T_1 = 5$ Ma, so the final error will be

$$d(T) = 5\delta_\nu \mathrm{e}^{(T-5)/5}. \tag{4.3}$$

One can look at this estimate in another way: the value of the precession frequencies is *ca.* $20''$ a^{-1}, which is *ca.* 10^{-4} rad a^{-1}. So the relative error of 10^{-6} gives

$$\delta_\nu = 10^{-4} \text{ rad Ma}^{-1}. \tag{4.4}$$

After 5 Ma, the error reaches 5×10^{-4} rad, and from equation (2.3) we get $T_V - 5$ Ma $\approx$ 33 Ma, i.e.

$$T_V \approx 38 \text{ Ma}, \tag{4.5}$$

which is much smaller than (3.1).

Table 2. *Change of precession of the planets due to the presence of the satellites*

	a (km)	m_S/m_P	$\Delta\dot{\varpi}$ (arcsec a^{-1})
Moon	383 398	1.230×10^{-2}	0.076 632
Callisto	1 883 000	5.668×10^{-5}	0.000 027
Ganymede	1 070 000	7.804×10^{-5}	0.000 012
Europe	670 900	2.527×10^{-5}	0.000 001
Io	421 600	4.705×10^{-5}	0.000 001
Titan	1 221 850	2.367×10^{-4}	0.000 006

5. Uncertainty of the orbital model

We have seen that the relative uncertainty of 10^{-6} in the planetary masses and initial conditions will already limit the time of validity of the solution to *ca.* 38 Ma, but this is without mentioning the errors coming from the modelling. Let us assume that the equations of motion are integrated exactly. We estimate here the effects not included in our model and which can reduce the time of validity of the solutions.

(*a*) *Effect of the satellites*

As was computed by Le Verrier (1858), the satellites of the planets induce a supplementary precession of their perihelion, which in a first-order computation gives

$$\left.\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right|_S = \frac{3}{4}\frac{m_0 m_1}{(m_0 + m_1)^2}\left(\frac{a}{a'}\right)^2 n',\tag{5.1}$$

where m_0, m_1 are the masses of the planet and the satellite, a, a' the semi-major axis of the satellite around the planet, and of the planet around the Sun, and n' is the mean motion of the planet. The values of these contributions for the large satellites of the Solar System are given in table 2. For the Moon, this first-order computation is not sufficient, and significant correction is due to higher-order terms. With a numerical integration of the Sun–Earth–Moon problem, we obtained $\Delta\dot{\varpi} = 0.065\,74''$ a^{-1}, while Bretagnon (1984) obtained $\Delta\dot{\varpi} = 0.065\,85''$ a^{-1} in his semi-analytical theory of the planets. This latter value was the one used in La93 (Laskar *et al.* 1993*a*).

The contribution of the Moon is already included in the orbital solution La93, but in order to get a more accurate solution over extended time, the higher-order part should be put in a completely analytical form. It can also be noted that the total contribution of the large satellites of the Solar System is $0.000\,047''$ a^{-1}. After 5 Ma, this will reach 10^{-3} rad, which will limit the validity of the solution to

$$T_V \approx 35\,\text{Ma}.\tag{5.2}$$

(*b*) *Tidal dissipation*

Due to the tidal dissipation in the Earth–Moon system, the Moon is receding at a rate of 3.82 cm a^{-1} (Dickey *et al.* 1994). This will thus change the value of the precession rate of the perihelion of the Earth–Moon barycentre given in table 2.

Table 3. *Change of precession of the planets due to the presence of small bodies*

	P (a)	m/m_0	$\Delta\dot\varpi$ (arcsec a^{-1})
Pluto	247.69	74.0×10^{-10}	0.11×10^{-6}
Ceres	4.6	5.9×10^{-10}	27.10×10^{-6}
Pallas	4.61	1.1×10^{-10}	5.03×10^{-6}
Vesta	3.63	1.2×10^{-10}	8.85×10^{-6}

From equation (5.1), we obtain the acceleration of the longitude of perihelion of the Earth due to this tidal dissipation as

$$\gamma_{\mathrm{M}} = \left.\frac{\mathrm{d}^2\varpi}{\mathrm{d}t^2}\right|_{\mathrm{M}} = 2\frac{\varpi}{a}\frac{\mathrm{d}a}{\mathrm{d}t}. \tag{5.3}$$

The error due to the omission of this contribution can be estimated, as previously, as

$$d(T) = \tfrac{1}{2}\gamma_{\mathrm{M}} \times T_1^2 \mathrm{e}^{(T-T_1)/5} \tag{5.4}$$

and its maximum value will be obtained for $T_1 = 10$ Ma, and reaches $d(T_1) \approx 700'' \approx 0.0035$ rad. This will limit the length of validity of the solution to

$$T_{\mathrm{V}} \approx 35 \text{ Ma}. \tag{5.5}$$

Of course, this contribution could be added to the model, but over the past 50 Ma there is uncertainty about the value of this tidal contribution, which is about one-third of the total contribution. In this case, this uncertainty will still limit the solution to

$$T_{\mathrm{V}} \approx 40 \text{ Ma}. \tag{5.6}$$

(*c*) *Effect of small bodies*

The contribution of a small planet of mass m' and mean motion n' can be given by a first-order approximation, which will give

$$\left.\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right|_A = \frac{3}{4}\frac{m'}{m_0 + m + m'}\left(\frac{n'}{n}\right)n' \tag{5.7}$$

and the perturbation of the longitude of the node is roughly the opposite of this value.

We can see here that Pluto has practically no effect on the Earth's orbit. This is not the case for the minor planets, the total contribution of which amounts to $41 \times 10^{-6''}$ a^{-1}. Moreover, the effect on Mars will be about double at *ca.* $77 \times 10^{-6''}$ a^{-1}, i.e. *ca.* 0.002 rad after 5 Ma, which will limit the validity of the solution to

$$T_{\mathrm{V}} \approx 32 \text{ Ma}. \tag{5.8}$$

(*d*) *Mass loss of the Sun*

The mass loss of the Sun due to solar radiation is

$$\frac{\dot{m}}{m} = -2.2 \times 10^{-21} \text{ s}^{-1}, \tag{5.9}$$

Table 4. *Change of precession of the planets due to general relativity for $t = J2000$*

(*J*2000 is the astronomical conventional origin of time. It is 1 January 2000 at 12 p.m. It also corresponds to the beginning of the Julian day 2451 545.0.)

	$\Delta\dot{\varpi}$ (arcsec a^{-1})
Mercury	0.429 805
Venus	0.086 246
Earth	0.038 387
Mars	0.013 509
Jupiter	0.000 623
Saturn	0.000 137
Uranus	0.000 024
Neptune	0.000 008

which leads to $\dot{m}/m = -0.35 \times 10^{-6}$ after 5 Ma. This will induce the same variation $\dot{a}/a = 0.35 \times 10^{-6}$ in the semi-major axis of the planets, which is below the present precision of the determination of the planetary initial conditions. The mass loss of the Sun is thus not an obstacle to the search for a planetary orbital solution over 50 Ma.

(*e*) *General relativity*

In the post-Newtonian formalism, the relativistic change of the perihelion velocity is

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t}\bigg|_R = \frac{3nG(M+m)}{a(1-\mathrm{e}^2)c^2} = \frac{3n^3a^2}{(1-\mathrm{e}^2)c^2}. \tag{5.10}$$

The computed values at the origin of time are given in table 4. It should be noted that this is uniquely the field due to the Sun. The field due to the other planets needs to be considered, but the effect of the planets on Mercury should be about 10^{-4} the value of the effect of the Sun. It should thus not be larger than $50 \times 10^{-6\prime\prime}\,\mathrm{a}^{-1}$ and thus of less importance than other sources of error, but it should still limit the solution to less than 35 Ma if it is not taken into consideration.

The terms of higher order in the general relativistic contribution will give to the previous expression a multiplying factor of the order of

$$\frac{GM}{ac^2} \approx 2.5 \times 10^{-8} \tag{5.11}$$

for Mercury, and are thus completely negligible.

(*f*) *Lens–Thirring effect*

This relativistic effect is due to the rotation of the Sun, which, according to Soffel (1989), amounts to $0.0001^{\prime\prime}\,\mathrm{a}^{-1}$ in the longitude of perihelion of Mercury. Neglecting it will limit the solution to 31 Ma.

Table 5. *Change of precession of the planets due to the effect of a solar $J_2 = 2 \times 10^{-6}$*

	$\Delta\dot{\varpi}$ (arcsec a^{-1})	$\Delta\dot{\Omega}$ (arcsec a^{-1})
Mercury	0.002 435	$-0.002\,435$
Venus	0.000 261	$-0.000\,261$
Earth	0.000 084	$-0.000\,084$
Mars	0.000 019	$-0.000\,019$

(g) J_2 of the Sun

The J_2 value of the Sun is not well known (see next section), but it is supposed to be small. Contrary to general relativity, the quadrupole moment of the Sun (J_2) affects both the longitude of node and perihelion (this is a way to discriminate between the two contributions). The rate of the perihelion due to the J_2 of the Sun is given by

$$\left.\frac{\mathrm{d}\varpi}{\mathrm{d}t}\right|_{J_2} = \tfrac{3}{4}J_2\left(\frac{R_\odot}{a}\right)^2 \frac{5\cos^2 i - 2\cos i - 1}{(1-\mathrm{e}^2)^2}n, \tag{5.12}$$

where $R_\odot$ is the radius of the Sun, n the mean motion of the planet, and i the inclination of the planet's orbit with respect to the equator of the Sun. The change in the rate of the node is

$$\left.\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right|_{J_2} = -\tfrac{3}{2}J_2\left(\frac{R_\odot}{a}\right)^2 \frac{\cos i}{(1-\mathrm{e}^2)^2}n, \tag{5.13}$$

which gives for the various planets the values given in table 5 for $J_2 = 2 \times 10^{-6}$. For this value of J_2, the precession due to the J_2 of the Sun is of $ca.\,0.003''\,\mathrm{a}^{-1}$. This would lead to an error of $ca.\,0.075$ rad after 5 Ma, which would limit the validity of the solution to

$$T_\mathrm{V} \approx 16 \text{ Ma}. \tag{5.14}$$

If the uncertainty on this term, which is the largest one considered so far, is reduced by a factor of 10 to 2×10^{-7}, the error would be 0.01 after 5 Ma, which will still limit the validity of the solution to

$$T_\mathrm{V} \approx 26 \text{ Ma}. \tag{5.15}$$

(h) Uncertainty in the measurement of the J_2 of the Sun

A precise measurement of the J_2 of the Sun has not yet been achieved. A compilation of some values in the literature gives until recently very large variations of the estimated values.

Campbell & Moffat (1983) Motion of the inner planets and Icarus:

$$J_2 = (5.5 \pm 1.3) \times 10^{-6}. \tag{5.16}$$

Landgraf (1992) Motion of Icarus:

$$J_2 = (0.6 \pm 5.8) \times 10^{-6}; \qquad J_2 < 2 \times 10^{-5}. \tag{5.17}$$

Paternó *et al*. (1996) Measure of the flattening of the Sun:

$$\approx 10^{-7} < J_2 < 5 \times 10^{-7}. \tag{5.18}$$

Bois & Girard (1998) Indirect measurement of the effect on the Moon's motion, for which very accurate observations are obtained with lunar laser ranging:

$$J_2 < 3 \times 10^{-6}. \tag{5.19}$$

Pijpers (1998) SOHO and GONG helioseismic data:

$$J_2 = (2.18 \pm 0.06) \times 10^{-7}. \tag{5.20}$$

Jurgens *et al*. (1998) Mercury radar ranging. Value of J_2 not yet determined but should be smaller than a few 10^{-7}.

These measurements appear to be quite different and not very conclusive, although they seem to converge to a value of a few 10^{-7}. The latest measurement of Pijpers (1998) is very precise, but it is not clear for me whether all the uncertainties of his model were taken into account in the computation of errors.

A direct measurement of the dynamical effect of the J_2 component of the Sun is thus welcome. The measurement of Bois & Girard (1998) through the lunar laser ranging data is interesting, but the intricate motion of the Moon, which is subjected to many other perturbations, should make it more difficult than direct measurements on Mercury. In this respect, as long as a drag-free solar probe is not used, the radar measurements made by Jurgens *et al*. (1998) should be at present the most effective way to obtain a direct determination of the J_2 of the Sun's gravity field, and in any case to bound its possible value.

6. First conclusions

Quite surprisingly, we have found that, at present, the main source of uncertainty for the construction of an accurate orbital solution for the Earth is the uncertainty in the determination of the J_2 value of the Sun. Even if there is some improvement in the near future, and if this error goes down to 10^{-7}, the validity time of the solution will be limited to 26 Ma. If this error decreases to 10^{-8}, the solution could be valid over 36 Ma, but we have seen that for this time span, there are several other sources of uncertainty which will limit the validity of the solution. At present, I would say that an attainable goal is to provide an accurate solution over 35 Ma. It can also be said that the present solution La93 is certainly not valid over more than 10–20 Ma, as was stated in Laskar *et al*. (1993*a*).

7. Precession and obliquity

Once the orbital solution of the Earth is known, one can compute the solution for the evolution of the Earth's precession and obliquity. The uncertainty resulting from this computation is of a different nature. Indeed, the motion of the obliquity is essentially stable, despite the proximity of a small resonance induced by the perturbation of Jupiter and Saturn (Laskar *et al*. 1993*a*, *b*). On the other hand, the rotational motion of the Earth is subject to various dissipative effects for which the amplitude and correct method of modelling are not known precisely (see Néron de Surgy & Laskar (1997) for a more complete review).

(*a*) *Change in the dynamical ellipticity of the Earth*

Although the possibility of changes in the dynamical ellipticity of the Earth was known for a long time, attention was drawn to it recently by Laskar *et al.* (1993*a*), when we demonstrated the proximity of the precession–obliquity solution of the Earth with a resonance with the $s_6 - g_6 + g_5$ term of perturbation due to Jupiter and Saturn. Although this excitation term is small, we demonstrated that it induces some important effects in the present solution of the obliquity of the Earth. Moreover, we showed that a very small change in the dynamical ellipticity of the Earth, of about 0.002 in relative size, could allow for a passage into resonance, thus inducing larger changes in the obliquity. In Laskar *et al.* (1993*a*), based on calculations by Thomson (1990), we mentioned that such a small change in the dynamical ellipticity of the Earth could be obtained by passage through an ice age, because of the change in the repartition of the mass loads on the Earth.

These findings were followed by new computations yielding improved estimates for the possible change in dynamical ellipticity when entering into an ice age (Peltier & Jiang 1994; Mitrovica & Forte 1995), which reveals the effects to be much smaller than the previous estimate used in Laskar *et al.* (1993*a*). With these new values, the passage into the resonance $s_6 - g_6 + g_5$ could no longer be obtained during an ice age. Nevertheless, the proximity of the resonance should still have a singular effect on the obliquity solution, and it should be noted that, due to the tidal evolution of the Earth–Moon system, we will surely enter into this resonance in the near future.

Recently, Forte & Mitrovica (1997) demonstrated that mantle convection could also have induced some small decrease in the dynamical ellipticity in the past, on a longer time-scale, which could reach about 0.01 within 20 Ma. They argued that this could allow again for a passage into the small Jupiter–Saturn resonance, but this is not clear since tidal evolution will have the opposite effect on this time-scale.

A different effect can also result from multiple passages into ice ages. Provided a certain time lag exists between the forcing of the obliquity and the ice age response (i.e. also the change in dynamical ellipticity), then a secular trend can occur in the variation of the obliquity of the Earth (Rubincam 1990, 1995; Bills 1994; Ito *et al.* 1995; Williams *et al.* 1998). Nevertheless, these effects occur only on very long time-scales, and their actual amplitude is still very controversial.

(*b*) *Tidal dissipation*

Due to the non-elasticity of the Earth, and to the fact that the Earth rotates faster on its axis than the Moon around the Earth, there will exist an offset between the tidal deformation of the Earth and the Earth–Moon direction. This induces a breaking couple on the rotation of the Earth, and by conservation of angular momentum, a slow increase in the Earth–Moon distance.

The first understanding of the tidal evolution of the Earth–Moon system was obtained by Darwin (1880), while modern developments are due to Kaula (1964), MacDonald (1964), Goldreich & Peale (1966), Goldreich (1966), Goldreich & Soter (1966), Lambeck (1979), Mignard (1979, 1980, 1981), Ward (1982), Laskar & Robutel (1993), Laskar *et al.* (1993*b*), Touma & Wisdom (1994) and Néron de Surgy & Laskar (1997).

The introduction of these tidal terms in the computation of the evolution of the precession and obliquity of the Earth over several millions of years was made in Quinn *et al.* (1991), Laskar *et al.* (1993*a*) and Néron de Surgy & Laskar (1997), while the resulting change in insolation was discussed in Berger *et al.* (1989).

In the La93 solutions, it was recognized that the uncertainty left in the value of the tidal dissipation, as well as the possible change of dynamical ellipticity, were the major sources of uncertainty for the precession and obliquity solution over 10–20 Ma. These two parameters were thus left free in the solutions, so that one could adjust them in the light of geological data. This was done in particular by Lourens *et al.* (1996).

(*c*) *Core–mantle interactions*

Another source of dissipation occurs at the core–mantle limit, due to the difference of the precessing rate of the core and the mantle (Rochester 1976; Goldreich & Peale 1966, 1967; Greenspan & Howard 1963; Lumb & Aldridge 1991; Néron de Surgy & Laskar 1997).

The exact value of this dissipation is largely unknown, as it depends on the effective viscosity of the outer core, but comparisons with available geological data for the evolution of the length of the day (Néron de Surgy & Laskar 1997) allow boundaries to be set on the possible value of this dissipation, which would not very much affect the solutions over 10–20 Ma. More precisely, since the value of the viscosity cannot be very large, it is difficult to discern the difference between a dissipation due to core–mantle interaction, and a tidal dissipation inducing the same effect on the breaking of the Earth's rotation (although the effects on the evolution of the obliquity are different).

(*d*) *Conclusions*

The uncertainty of the dissipative effects due to tidal dissipation, core–mantle interactions, and changes in dynamical ellipticity are real, but if the geological data are precise enough, this should not be a real problem for the orbital solution. Indeed, the behaviour of obliquity is stable, and fitting the dissipative contribution to the geological data should be possible, as has already been done in Lourens *et al.* (1996).

8. Beyond chaos

In the previous sections we have seen that in order to obtain an accurate solution for the orbital motion, we are practically limited to 35 Ma, due to the exponential divergence of the solutions. On the other hand, the dissipative effects present a large uncertainty, but they can be adjusted in the light of the geological data. The question which remains is how to cope with the chaos, or more precisely, is it possible without pretending to have an accurate solution for the Earth to still get some information in an astronomical solution to use to obtain a geological time-scale over much longer times, extending over 250 Ma, which correspond roughly to the time-scale for which geological data are available?

In order to address this problem, one needs to look more closely at the expected signal which can be extracted from the geological data.

Table 6. *Principal terms in the astronomical solution La93(1,0) analysed over* 4 Ma

(It should be noted that the values of the frequencies and periods are given here only as internal indications. As they are determined over only 4 Ma, they are not as accurate as the ones provided in Laskar (1990), especially for the $g_2 - g_5$ component.)

	$e = \sum_i a_i \cos(\nu_i t + \phi_i)$			
ν	ν_i (arcsec a^{-1})	period (a)	a_i	ϕ_i (deg)
	0.0000		0.028 159	
$g_2 - g_5$	3.1906	406 182	0.010 851	168.02
$g_4 - g_5$	13.6665	94 830	0.009 208	121.71
$g_4 - g_2$	10.4615	123 882	0.007 078	−36.79
$g_3 - g_5$	13.1430	98 607	0.005 925	−86.17
$g_3 - g_2$	9.9677	130 019	0.005 295	115.55
	$\epsilon = \sum_i a_i \cos(\nu_i t + \phi_i)$			
ν	ν_i (arcsec a^{-1})	period (a)	a_i	ϕ_i (deg)
	0.0000		0.406 657	0.00
$p + s_3$	31.6132	40996	0.011 168	86.26
$p + s_4$	32.6799	39657	0.004 401	89.28
$p + s_3 + g_4 - g_3$	32.1827	40270	0.003 010	103.74
$p + s_6$	24.1277	53714	0.002 912	−46.03
$p + s_3 - g_4 + g_3$	31.0981	41674	0.002 625	−117.31
$p + s_1$	44.8609	28889	0.001 452	−175.78
	$e \sin \varpi = \sum_i a_i \cos(\nu_i t + \phi_i)$			
ν	ν_i (arcsec a^{-1})	period (a)	a_i	ϕ_i (deg)
$p + g_5$	54.7064	23690	0.018 839	−52.51
$p + g_2$	57.8949	22385	0.016 981	113.26
$p + g_4$	68.3691	18956	0.014 792	71.29
$p + g_3$	67.8626	19097	0.010 121	−131.00
$p + g_1$	56.0707	23114	0.004 252	−3.96

(*a*) *The astronomical solution*

In the sedimentary data, it is not unusual to detect the signature from eccentricity e, obliquity ε, and climatic precession $e \sin(\hat{\varpi})$, where $\hat{\varpi}$ is the longitude of perihelion from the moving equinox. In table 6, the first terms of these quantities, obtained by frequency analysis from the La93(1,0) nominal solution, are given, together with their correspondence as a combination of the fundamental secular frequencies g_i, s_i of the Solar System, and of the precession frequency p. The reader should refer to Laskar (1988, 1990) and Laskar *et al.* (1993*a*) for more details.

It should be remembered that the motion of the outer planets is very stable. This is also the case for their corresponding secular frequencies. Indeed, in Laskar (1990) it was shown that $g_5, g_6, g_7, g_8, s_6, s_7, s_8$ are practically constant over 200 Ma. The associated argument can most certainly be used for establishing the time-scale of the geological data, provided its signature actually appears in the data.
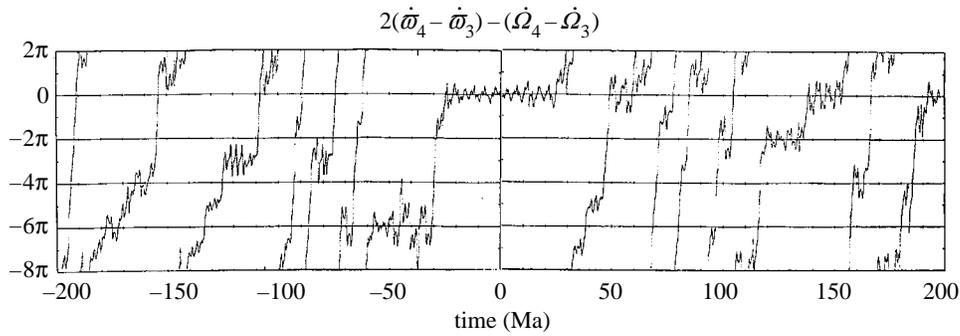
$$2(\dot{\varpi}_4 - \dot{\varpi}_3) - (\dot{\Omega}_4 - \dot{\Omega}_3)$$

Figure 8. Change from libration to circulation in the argument of the $(s_4 - s_3) - 2(g_4 - g_3)$ resonance (Laskar 1992).

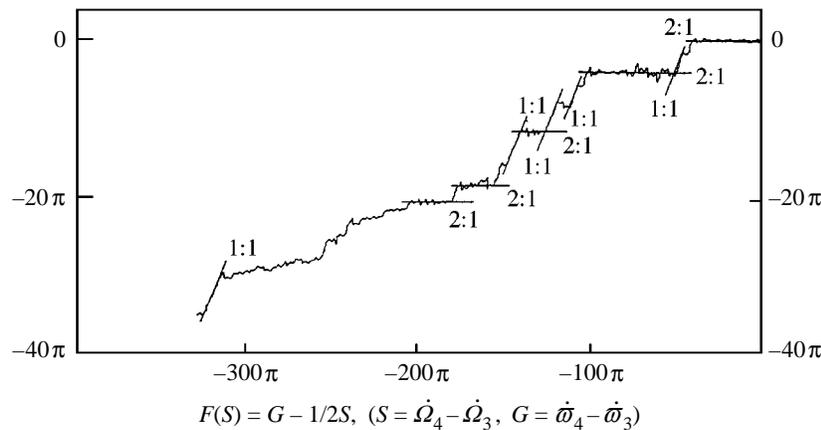$$F(S) = G - 1/2S, \quad (S = \dot{\Omega}_4 - \dot{\Omega}_3, \ G = \dot{\varpi}_4 - \dot{\varpi}_3)$$

Figure 9. Transition from the 1:2 secular resonance $((s_4 - s_3) - 2(g_4 - g_3))$ to the 1:1 resonance $((s_4 - s_3) - 2(g_4 - g_3))$, as observed in the numerical integration over 200 Ma in the past (Laskar 1992).

On the contrary, the arguments associated with g_1, g_3, g_4, s_1, s_2, s_3, s_4 are quite unstable (Laskar 1990). The last frequency g_2 is moderately unstable, but the behaviour of its associated angle needs to be studied further over extended time-scales.

This frequency is very interesting, as the $g_2 - g_5$ terms appear with a period of *ca.* 406 ka and large amplitude in the eccentricity solution, and also in the modulation of the amplitude of the 22 ka term, due to the beat between the $p + g_5$ and $p + g_2$ terms in the climatic precession $e \sin \hat{\varpi}$. For establishing long time-scales, this term should be preferred to the other ones involving either unstable frequencies, $g_1, g_3, g_4, s_1, s_2, s_3, s_4$, or the precession frequency p, the evolution of which will depend very much on dissipative effects, which are not so well known.

9. Detection of chaos in the geological data

A very important observation can be made by looking at table 6: in the obliquity signal, the two terms $p + s_4$ and $p + s_3$ appear with very large amplitudes. They

will thus induce a modulation in the amplitude of the 40 ka signal, with frequency of $s_3 - s_4 \approx 1.0667'' \, \mathrm{a}^{-1}$ (period *ca.* 1.215 Ma). Such a signal seems to be present in the ODP 154 record (see Shackleton *et al.*, this issue).

On the other hand, in the climatic precession, the two terms $p+g_4$ and $p+g_3$ should induce also a modulation of frequency $g_4 - g_3 \approx 0.5236'' \, \mathrm{a}^{-1}$ (period *ca.* 2.475 Ma) in the 19 ka term of the climatic precession, as well as in the 95 and 125 ka terms in the eccentricity. For these two last terms, it should be noted that even if the resolution of the data does not make it possible to discriminate between the 95 and 125 ka terms, the modulation of the amplitude of these terms is the same, and thus could still be discernible in the geological record.

If it were possible to obtain these two modulations from the geological data, this would have important consequences. Indeed, in Laskar (1990) it was demonstrated that the resonance

$$(s_4 - s_3) - 2(g_4 - g_3) = 0 \tag{9.1}$$

is one of the main sources of the chaotic behaviour in the motion of the planets. Moreover, I could show that presently we are in a librational state with respect to this resonance, but this can evolve in a rotational state, and even slowly move to libration in a new resonance, namely

$$(s_4 - s_3) - (g_4 - g_3) = 0. \tag{9.2}$$

The transition from this 1:2 resonance of $s_4 - s_3$ and $g_4 - g_3$ to the 1:1 resonance should be possible to detect. If this is the case, this would be the signature of the chaotic motion of the planets. Moreover, the dating, even in a very approximate manner of these transitions, would provide very precise constraints on the dynamical model for the evolution of the Solar System. It would be even more important to find the first transition in the past, as this would be the event which could be actually used for adjusting (or at least testing) the parameters of the orbital solution. One should realize that in this case the exponential divergence of the solution will be used in the reverse way to that in which it was used in the first sections of this paper, and could make it possible to obtain very precise information on the initial conditions and (or) parameters of the model. One could even dream that if the succession of the transitions from the 1:2 to the 1:1 resonance were found and dated over an interval of 200 Ma that this could be the ultimate test for the gravitational model. It would make it possible, for example, to obtain the J_2 value of the Sun with high accuracy, or to test the model of general relativity.

10. Conclusion

Due to the chaotic behaviour, the time of validity for a precise orbital solution of the Earth will be in practice limited to 35–50 Ma. Moreover, one of the main sources of uncertainty at present is the imprecision in the measurement of the J_2 value of the Sun which could even bring this limit down to a much shorter time span. Nevertheless, it can be forecast that within five years much more accurate knowledge of this quantity should be obtained. In this case, there are still numerous sources of uncertainty which will limit the solution to 35–50 Ma.

There are also several sources of uncertainty for the dissipative effects in the evolution of the rotational and precession motion of the Earth, but comparison with

the geological data should permit adjustment of these quantities. It should also be possible to use the astronomical time-scale beyond the limit of 35–50 Ma if one acknowledges the unavailability of an accurate solution for the orbital motion of the Earth, but searches in the data for the signature of frequencies related to the motion of the outer planets, which is predictable over much longer time-scales.

Moreover, the $s_4 - s_3$ and $g_4 - g_3$ frequencies induce some modulation in the amplitude of the obliquity and eccentricity or climatic precession. It should thus be possible to observe in the geological data the trace of transition from the $(s_4 - s_3) - 2(g_4 - g_3)$ secular resonance to the $(s_4 - s_3) - (g_4 - g_3)$ resonance. The detection and dating of these passages, which are the signature of the chaotic behaviour of the planets, should induce extremely high constraints on the dynamical models for the orbital evolution of the Solar System. This gives a unique and challenging opportunity for palaeoclimate records to provide some of the ultimate constraints on the dynamical models for the evolution of the Solar System.

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