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RANDOM FLUCTUATIONS, PERSISTENCE, AND QUASI-PERSISTENCE IN GEOPHYSICAL AND COSMICAL PERIODICITIES

BY J. BARTELS

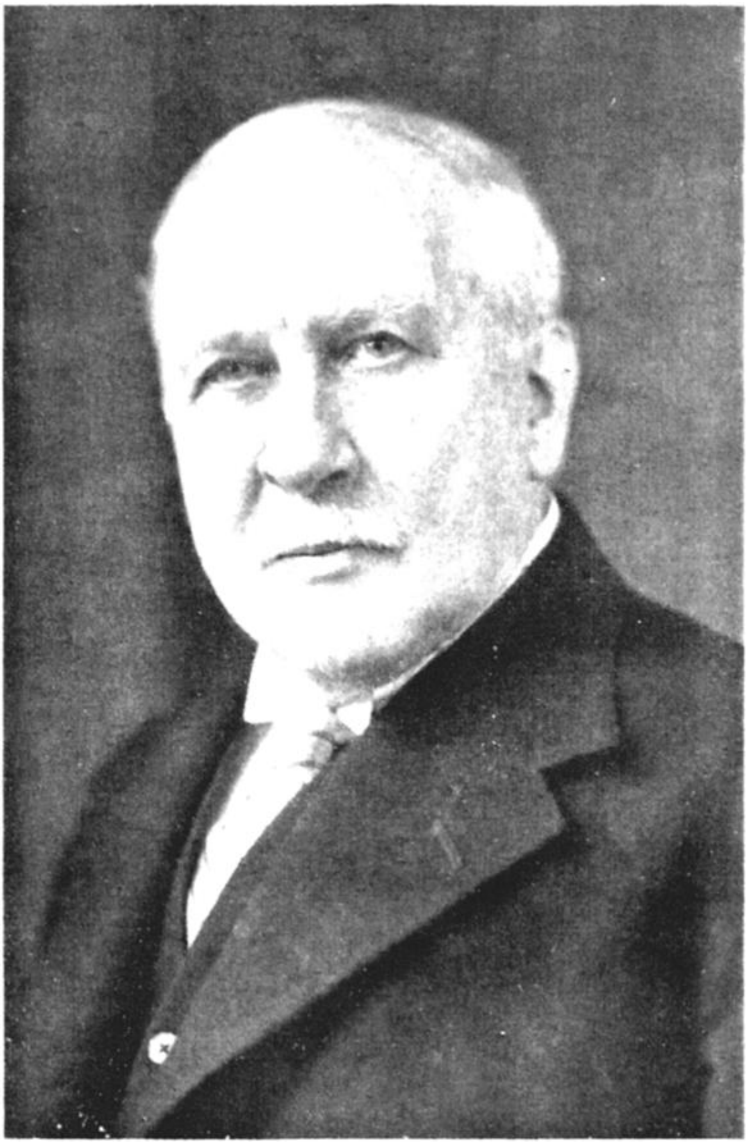
Abstract—The statistical aspects of the application of harmonic analysis, introduced by A. Schuster in his famous paper on the investigation of hidden periodicities, are discussed on the basis of recent developments in the theory of probability. Between the two extreme cases of random fluctuations and persistent waves, hitherto discussed exclusively, the intermediate case of quasi-persistence is introduced and recognized as a common phenomenon in the time-functions of meteorology, geophysics, and cosmical physics. Statistical methods, based on the conception of the harmonic dial, are given for dealing with quasi-persistence and its effect on tests for persistent waves, and they are generalized for the case of periodicities of other form than that of the sine-wave. Typical examples are given illustrating various forms of random fluctuations, quasi-persistence, and persistence, as well as questions related to harmonic analysis, such as the periodogram, non-cyclic change, curvature-effect, equivalent length of sequences, effective expectancy, random walk, interference, and the infective property of quasi-persistence on adjacent periods (see summary at end of paper).

(I) INTRODUCTION

1. *The problem*—Investigations on periodicities, cycles, recurrence-tendencies, and similar phenomena in geophysics proceed, in general, in three stages: (1) Analytical transformations of the observational data, for instance, harmonic analysis; (2) statistical studies on the results of these transformations, testing the degree of their significance; (3) physical explanations of the significant periodicities, for instance, by rotation-periods of the celestial bodies, by free or forced wave-motions or oscillations, etc.

These three stages are not in every case of equal importance, nor is their order invariable. Tidal theory, for instance, starts from the well-known movements of the celestial bodies and develops a specially adapted harmonic analysis, and there is hardly a need for the statistical viewpoint. The situation is, however, different with respect to the great number of geophysical periodicities in which the length of the period is given beforehand and only the form of the actual periodic variation in this interval is wanted, for instance, in the case of the solar and lunar diurnal variations and the annual variations, which occur in practically every geophysical phenomenon. Here statistical methods have been applied successfully, for instance, in the case of the solar diurnal magnetic variations, which show a marked day-to-day variability,[1,2] or in the case of the lunar diurnal variations of terrestrial magnetism or of atmospheric pressure, where small periodic changes are masked by much larger time-changes of different character, or in the case of the semi-

[1] S. Chapman and J. M. Stagg, London, Proc. R. Soc., A, 123, 27-53 (1929); 130, 668-697 (1931).
[2] J. Bartels, Terr. Mag., 37, 291-302 (1932).



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annual variation of magnetic activity. There is a large and promising field for further use of such methods.

It seems, however, even more urgent to improve the procedure for testing the significance of such periodicities in which not even the lengths of the periods or recurrence-intervals are known or suspected from the outset. A large number of periods and cycles has been claimed in atmospheric temperature, rainfall,[3] solar radiation,[4] earthquakes, and even in business-activity,[5] while only very few of them have been generally recognized. This strange result has brought about a state of uncertainty and instinctive distrust which sometimes even affects the attitude towards perfectly sound periodicities. An attempt is made in this paper to discuss the elementary principles underlying research of periodicities. The main reasons for the contradictory results will be found in the lack of adequate combination of harmonic analysis with the theory of probability in its modern form.[6]

2. *Schuster's periodogram*—The "Investigation of hidden periodicities" published in 1898 by A. Schuster in this JOURNAL[7] has become famous because it is the first successful attempt to "introduce a little more scientific precision into the treatment of problems which involve hidden periodicities" by applying the theory of probability. A. Schuster calculated his "periodogram" for 25 years of records of magnetic declination at Greenwich[8] and for sunspot-data,[9] modifying his original method according to the optical analogy between the periodogram and the spectrum of a luminous disturbance. A number of periodograms have been calculated since then; considerable progress in the practical application of the Schuster method, speeding up the heavy arithmetical work connected with it, has been made by K. Stumpff,[10] using instrumental methods, and L. W. Pollak,[11] who analyzed the international magnetic character-figure (designated C in this paper) for the years 1906 to 1926 using punched cards and Hollerith tabulating machines.

3. *A short review of literature*—Since Schuster's papers were written, a number of investigations in pure mathematics and theoretical physics have appeared bearing on subjects which are connected with periodogram-analysis—though this connection is not expressly mentioned and, sometimes, not even realized. Since these studies may be utilized for a revision and development of the periodogram-method, some of them may be enumerated here. On the analytical side, the theory of "almost periodic functions" created by H. Bohr[12, 13] generalizes the ordinary Fourier series by considering sums of sine-waves with frequencies which

[3]See, for instance, the puzzling list of periods ranging from a few hours to 260 years in Sir Napier Shaw's Manual of Meteorology, vol. 2, pp. 312-327, Cambridge, 1928.

[4]C. G. Abbot, Smithson, Misc. Coll., **87**, No. 9 (1932); **87**, No. 18 (1933); **89**, No. 5 (1933).

[5]Edwin B. Wilson, Science, **80**, 193-199 (1933); Quart. J. Economics, 375-417 (May 1934).

[6]R. von Mises, Wahrscheinlichkeitsrechnung, Leipzig und Wien, 1931; E. Kamke, Einführung in die Wahrscheinlichkeitstheorie, Leipzig, 1932; A. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung, Ergebn. Math., **2**, Nr. 3, Berlin (1933); A. Khintchine, Asymptotische Gesetze der Wahrscheinlichkeitsrechnung, Ergebn. Math., **2**, Nr. 4, Berlin (1933).

[7]Terr. Mag., **3**, 13-41 (1898).

[8]Cambridge, Phil. Trans., **18**, 107-135 (1899).

[9]London, Phil. Trans. R. Soc., A, **206**, 69-100 (1906).

[10]K. Stumpff, Analyse periodischer Vorgänge, Berlin, 1927.

[11]L. W. Pollak, Prager Geophysikalische Studien, Heft 3 (Čechoslovak. Statistik, Reihe 12, Heft 13), Prague, 1930.

[12]Harald Bohr, Fastperiodische Funktionen, Ergebn. Math., **1**, Nr. 5, Berlin (1932).

[13]A. S. Besicovitch, Almost periodic functions, Cambridge, 1932; N. Wiener, The Fourier integral and certain of its applications, Cambridge, 1933.

are *not* entire multiples of a fundamental frequency. On the statistical side, the fundamental problem variously named "random vibrations," "random flights," or "random walk" (Irrfahrt), has been treated by Lord Rayleigh,[14] on whose first paper A. Schuster based his periodogram, J. C. Kluyver,[15], K. Pearson,[16], G. Pólya,[17] and G. I. Taylor.[18] Some relations can also be found to papers on Brownian movement, or eddy-diffusion in the atmosphere.[19, 20, 21] A part of the optical analogy, the superposition of light-waves with random phases, has been treated by M. von Laue[22] and A. Einstein.[23] A. Basch's theory of "error-tensors"[24] developed for geodetical purposes must also be mentioned. A. Glogowski,[25] in a dissertation on hidden periodicities, does not sufficiently emphasize the statistical viewpoint and misconstrues Schuster's methods.

Of the many papers dealing with geophysical and cosmical periodicities, a few may be selected as containing theoretical discussions of the periodogram-method. G. U. Yule[26] discusses the effect of superposed fluctuations and disturbances on harmonic analysis. Sir Gilbert Walker[27] defines criteria for reality of periods. L. Weickmann's discovery of "symmetry-points" in the records of atmospheric pressure entailed a number of studies on periodicity in general.[28, 29] H. H. Turner[30] considered discontinuities in meteorological phenomena. Leo Keller[31] amplifies the mathematical system of periodography in a form suitable for geophysical applications.

4. *Plan of this paper*—It is not proposed to give here a bibliographical account of the contributions of the various authors to the theory of periodogram-analysis. It seems to be more convenient to derive the new results directly by using elementary graphical illustrations of harmonic analysis.

It would have been possible to derive the results of this paper in a quite general way, discussing mathematical-statistical properties of "populations" formed by a number of vectors in two or more dimensions. However, it seemed more appropriate to show the need for these considerations by dealing with time-functions representing actual geophysical phenomena. After the introduction of the conception of persistence and quasi-persistence as contrasted with random fluctuations,

[14] Lord Rayleigh, Phil. Mag., **10**, 73-78 (1880); **36**, 429-449 (1918); **37**, 321-347, 498-515 (1919). Reprinted in Scient. Papers **1** and **6**, Cambridge, 1899 and 1920.

[15] J. C. Kluyver, Amsterdam, Proc. Akad. Wet., **8**, 341-350 (1906).

[16] K. Pearson, A mathematical theory of random migration (Math. contrib. to the theory of evolution, 15), London, 1906.

[17] G. Pólya, Zürich, Mitt. Physik. Ges., **19**, 75-86 (1919).

[18] G. I. Taylor, London, Proc. Math. Soc., **20**, 196 ff. (1922).

[19] O. G. Sutton, London, Proc. R. Soc., A, **135**, 143-165 (1932).

[20] L. F. Richardson and J. A. Gaunt, London, Mem. R. Met. Soc., **3**, No. 30 (1930).

[21] O. F. T. Roberts, London, Mem. R. Met. Soc., **4**, No. 37 (1933).

[22] M. von Laue, Ann. Physik, **47**, 853-878 (1915); **48**, 668 ff. (1915).

[23] A. Einstein, Ann. Physik, **47**, 879-885 (1915).

[24] Wien, SitzBer. Akad. Wiss., Math.-Nat. Klasse, Abt. IIa, **137**, 583-598 (1928).

[25] A. Glogowski, Beiträge zur Auffindung verborgener Periodizitäten, Münster i. W., 1929.

[26] G. U. Yule, London, Phil. Trans., A, **226**, 267-298 (1927).

[27] Sir Gilbert Walker, London, Quart. J. R. Met. Soc., **51**, 337-346 (1925); London, Mem. R. Met. Soc., **1**, No. 9 (1927); **3**, No. 25 (1930); Mon. Weath. Rev., **59**, 277-278 (1931); London, Proc. R. Soc., A, **131**, 518-532 (1931). See also D. Brunt, Memoirs R. Met. Soc., **2**, No. 15 (1928), the discussion in London, J. R. Met. Soc., **54**, 299-303 (1928), and R. A. Fisher, London, Proc. R. Soc., A, **125**, 54-59 (1929).

[28] L. Weickmann, Beitr. Geophys., **34**, 244-251 (1931).

[29] K. Stumpff, Beitr. Geophys., **32**, 379-411 (1931); F. Dilger, Beitr. Geophys., **30**, 40-95 (1931).

[30] H. H. Turner, London, Quart. J. R. Met. Soc., **41**, 315-336 (1915); **42**, 163-173 (1916); **43**, 43-60 1917).

[31] L. Keller, Beitr. Physik frei. Atmos., **19**, 173-187 (1932).

if often appeared unnecessary to repeat the definitions in general abstract formulations. Furthermore, we regard throughout the paper all time-functions as given by values at equidistant intervals of time. This assumption clarifies the argument and holds in most geophysical applications. Continuous functions of time could have been treated in exactly the same way, replacing the sums by integrals, without introducing a fundamentally different conception; in fact, continuous recording is practically represented by values at very short intervals of time.

The standpoint taken in the present paper is the outcome of work on periods in meteorology and terrestrial magnetism, and has been discussed during several years in a number of talks at the Department of Terrestrial Magnetism of the Carnegie Institution of Washington and in courses of lectures at Berlin University. Exact proofs for some theorems involving theory of probability are omitted here and will be given in a later paper, to appear in the series "Ergebnisse der Mathematik" (Berlin, J. Springer).

(II) Harmonic Analysis as a Mathematical Representation of the Observations

5. *Principle of harmonic analysis*—Records of geophysical phenomena yield functions of time, f(t), which for further research are mostly transformed into a series of values for equal intervals of time, for instance, hourly, daily, monthly, annual, etc. The record may cover the time $t=0$ to $t=T$. For convenience, another time-variable, $x=t\times 2\pi/T$, is introduced so that the length of the record, as measured by x, is 2π. The number of values (or ordinates) given may be r; that is, the times (or abscissae) $x_1, x_2, \ldots x_r$ divide the time-interval 0 to 2π into r equal parts, and y_p may be the value of the variable for the time

$$x_p = p(2\pi/r) \tag{5.1}$$

No attention is paid, at this stage, to the value y_0 at the time $x_0=0$ (see section 16).

Consider sine-functions and cosine-functions of frequency $\nu=0, 1, 2, \ldots k$, that is, completing ν cycles in the interval 0 to 2π [lengths of periods $p_\nu=T/\nu$], and their sum

$$\begin{aligned}\phi_k(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)\\ +\ldots+(a_k \cos kx + b_k \sin kx)\end{aligned} \tag{5.2}$$

Harmonic analysis consists in determining the coefficients $a_0, a_1, b_1, \ldots a_k, b_k$ so that $\phi_k(x_p)$ approximates the given ordinates y_p, in other words, that the *residuals* $[y_p-\phi_k(x_p)]$ are as small as possible. This problem is readily solved if it is put into the form that the average of the squared residuals

$$s_k^2 = \Sigma_p\, [y_p - \phi_k\,(x_p)]^2/r \tag{5.3}$$

shall be made a minimum.

Since ϕ_k contains $(2k+1)$ coefficients and shall represent r ordinates, we consider only values of k so that

$$2k+1 \leqq r \tag{5.4}$$

Then it can be shown that the coefficients are given by the equations

$$r\, a_0 = \Sigma_p\, y_p,\quad (r/2)\, a_\nu = \Sigma_p\, y_p \cos \nu x_p, \text{ and } (r/2)\, b_\nu = \Sigma_p\, y_p \sin \nu x_p \tag{5.5}$$

where the sums are taken for $\rho=1, 2, \ldots r$, and ν runs from 1 to k. a_0 is the arithmetic mean of the y_ρ, and (a_ν, b_ν) are called the harmonic coefficients of the set y_ρ of ordinates. If r is an even number, the formula for $a_{r/2}$ is

$$(5.6) \qquad a_{(r/2)}=(-y_1+y_2-y_3+y_4-\ldots+y_r)/r$$

which differs from the formula for $\nu<r/2$ in so far as the right-hand sum is divided by r and not by $(r/2)$.

From the linear form of the equations (5.5), the theorem on *superposition* of different functions is easily verified, namely: A finite number of ordinates $y'_1, y'_2, \ldots y'_r; y''_1, y''_2, \ldots y''_r; \ldots$ may be given, and (a'_ν, b'_ν) may be the harmonic coefficients for the set $y'_1, y'_2, \ldots y'_r$, etc. Then a set of ordinates formed by the linear combination $A'y'_\rho+A''y''_\rho+\ldots$ $(\rho=1, 2, \ldots r)$, with constants $A', A'', \ldots$, has the harmonic coefficients $(A'a'_\nu+A''a''_\nu+\ldots, B'b'_\nu+B''b''_\nu+\ldots)$. This is known as the *additive property of the harmonic coefficients*, or principle of *superposition*.

The formulae (5.5) do not contain any reference to k, that is, the number of terms of the series $\phi_k(x)$. Each harmonic coefficient is therefore determined *independently*, regardless of the number of additional harmonic terms involved. This is a consequence of the so-called *orthogonality* of sine-waves and cosine-waves with periods which are submultiples of one and the same main period.

A proof of the formulae (5.5), and a discussion of some other points such as smoothing, non-cyclic variation, etc., is given in the *appendix*.

6. *The harmonic dial*—The sine- and cosine-functions of frequency ν can be combined into a *sine-wave* with (positive) amplitude c_ν and phase α_ν

$$(6.1) \qquad a_\nu \cos \nu x+b_\nu \sin \nu x=c_\nu \sin (\nu x+\alpha_\nu), \text{ with}$$

$$(6.2) \qquad a_\nu=c_\nu \sin \alpha_\nu,\ b_\nu=c_\nu \cos \alpha_\nu \text{ and } c_\nu^2=a_\nu^2+b_\nu^2,\ \tan \alpha_\nu=a_\nu/b_\nu$$

These relations can be illustrated in the *harmonic dial* for the frequency ν. In a plane coordinate system, in which a_ν is measured upward, and b_ν to the right (Fig. 1), the expression (6.1) is represented by a point P

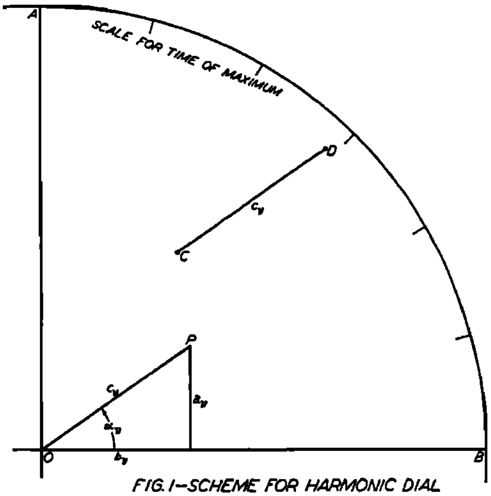


FIG. 1—SCHEME FOR HARMONIC DIAL

having the rectangular coordinates a_ν, b_ν, and, because of (6.2), the polar coordinates (c_ν, α_ν); or, also, by the vector OP having the projections a_ν, b_ν on the axes, the length c_ν and the azimuth α_ν. This vector will be called $\mathbf{c}_\nu$. The first of the ν maxima of the wave (6.1) occurs when $(\nu x+\alpha_\nu)=90°$, that is, at the time $x_{max}=[90°-\alpha_\nu]/\nu$. Therefore, $\alpha_\nu=90°$ corresponds to $x_{max}=0$, $\alpha_\nu=0°$ to $x_{max}=90°/\nu$, etc. It is therefore possible to indicate, on a circle around the origin, the times x_{max} (or t_{max}, expressed in the original time t) for the waves represented by vectors pointing in that direction. This gave the name to the diagram, because, in a semi-diurnal wave [time interval T from 0^h to 12^h], $t=1^h$ corresponds to $x=2\pi/12=30°$, and the scale for t_{max} becomes the ordinary dial of a clock.

The "blank" for a harmonic dial of a certain frequency contains the origin O, a linear scale for the amplitudes c_ν [or a number of circles around the origin designating certain values of c_ν] and a circular scale at the edge, marked with the occurrence of the maximum and, incidentally, giving the length of the period p_ν. Changes of units for c_ν or of time origin [for instance, from local to Greenwich time in dials for diurnal waves] are easily indicated by renumbering the respective scales. Each point P entered, as a dot, in this blank represents, by the vector OP, a sine-wave of the period p_ν. Since the blanks for harmonic dials of the period p_ν for the intervals $t=0$ to T, T to $2T$, $2T$ to $3T$, etc., are identical except for the numbering of the circular scales, which differ by multiples of T, they can all be combined into that for the interval $t=0$ to T, because the various intervals can be indicated by marking the dots P.

7. *Vector-addition in harmonic dials and the average vector*—It is sometimes convenient to ascribe to each vector CD on the harmonic dial the same meaning as to the parallel vector OP starting at the origin, so that all parallel vectors of equal length denote the same sine-wave. Then, the additive property of the harmonic coefficients [section 5] has its graphical analogy in the usual *vector-addition*.

A number (say, n) of sine-waves of equal frequency ν may be indicated as vectors $\mathbf{c}'_\nu$, $\mathbf{c}''_\nu$, ... starting at the origin O, and plotted as dots denoting the ends of the vectors. If these sine-waves are added and divided by n, the average sine-wave has the harmonic coefficients $[(a'_\nu+a''_\nu+\ldots)/n]$, $[(b'_\nu+b''_\nu+\ldots)/n]$ and is therefore represented by the mass-center of the n dots, or the *average vector* $(\mathbf{c}'_\nu+\mathbf{c}''_\nu+\ldots)/n$.

This remark is often used as follows: Suppose the number r of ordinates is an entire multiple of the frequency ν, say, $r=\nu r_1$. Then the angles νx_ρ (5.1) are $\nu\rho\,(2\pi/\nu r_1)=\rho(2\pi/r_1)$, so that (apart from irrelevant multiples of 2π), $\nu x_1=\nu x_{r_1+1}=\nu x_{2r_1+1}=\ldots$, etc. The equation (5.5) for a_ν (and for b_ν) can therefore be rearranged as follows

$$
\begin{aligned}
(7.1)\qquad a_\nu=(2/r)\,\Sigma_{\rho=1}^{r}\,y_\rho\cos\nu x_\rho=1/\nu\,&[(2/r_1)\,\Sigma_{\lambda=1}^{r_1}\,y_\lambda\cos\lambda\,(2\pi/r_1)\\
&+(2/r_1)\,\Sigma_{\lambda=1}^{r_1}\,y_{r_1+\lambda}\cos\lambda\,(2\pi/r_1)+\ldots\\
&+(2/r_1)\,\Sigma_{\lambda=1}^{r_1}\,y_{(\nu-1)r_1+\lambda}\cos\lambda\,(2\pi/r_1)]
\end{aligned}
$$

Comparing the first term in the bracket with (5.5), we realize that it is the coefficient for frequency 1 of the ordinates y_1 to y_{r_1}, and the second term is the coefficient for frequency 1 of the ordinates y_{r_1+1} to y_{2r_1}, etc., and a_ν is the average of these ν coefficients. In other words, if a period p comprises an interval represented by r_1 ordinates, and ν such intervals are given, then the harmonic analysis of the total of νr_1 ordinates gives, for the period p, harmonic coefficients which are the arithmetic means of the ν harmonic coefficients computed from each single interval of r_1 ordinates.

In another arrangement, (7.1) becomes

$$a_\nu=(2/r_1)\ \Sigma_{\lambda=1}^{n}\ (1/\nu)\ (y_\lambda+y_{r_1+\lambda}+y_{2r_1+\lambda}+\ldots+y_{(\nu-1)r_1+\lambda})\ \cos\lambda\ (2\pi/r_1) \tag{7.2}$$

This is the basis of many schemes for numerical harmonic analysis, starting by writing the ordinates in ν rows of r_1 each, and then analyzing the averages of the r_1 columns.

8. *International magnetic character-figure C and harmonic dial for 27-day period*—Examples demonstrating the use of the harmonic dial for research on solar and lunar diurnal variations and for annual variations have been given formerly.[32] For the purpose of this paper, the series of the daily international magnetic character-figures C has been selected, comprising the 10,206 days between January 11, 1906, and December 20, 1933. C indicates the degree of magnetic activity for each Greenwich day by one of the figures 0.0 (denoting very quiet conditions), 0.1, 0.2, etc., to 2.0 (denoting very great disturbances). The rotation-period of the Sun, of about 27 days, is reflected in C in the recurrence of quiet and disturbed times.[33] This recurrence is demonstrated in graphical day-by-day records published in this JOURNAL.[34] For these diagrams, the whole series has been divided into 27-day intervals. For convenience, we shall refer to these intervals as "rotations" numbered 1 (beginning January 11, 1906) to 378 (beginning November 24, 1933). In each rotation the days are numbered 1 to 27. The dates of the first days in each rotation can be taken from the diagram in Volume 39 of this JOURNAL or from the table on Figure 15 of this paper; the dates are repeated, with a shift of one or two days (after leap-years), every second year, since $2\times365=27\times27+1$.

The character-figures C for the years 1906 to 1926 have been used in Pollak's publication.[11] It may be remarked, however, that it is not intended here to demonstrate again the 27-day recurrence or to repeat Pollak's periodogram-analysis: the series of C is only taken as a suitable illustration of the general argument, which will gradually lead to other conclusions than those drawn by Pollak.

For each of the 378 rotations, the harmonic coefficients of the sine-wave of 27-day period were computed and the results are represented in the harmonic dial of Figure 2. The dots are distributed in a "cloud" around the origin without, apparently, preferring any direction: the average vector, that is, the mass-center of the cloud formed by all dots, indicated by a cross, falls close to the origin. The largest amplitude is 0.760 unit of C [or 0.760C] for rotation No. 208, beginning May 1, 1921, and containing the heaviest magnetic disturbances [about May 12

[32] J. Bartels, Zs. Geophysik, **3**, 389-397 (1927); Handbuch d. Experimentalphysik, **25**, I. Teil, 167 ff., 631 ff. (Leipzig, 1928); Sci. Mon., **35**, 110-130 (1932); Terr. Mag., **37**, 22-27, 291-302 (1932).

[33] C. Chree and J. M. Stagg, London, Phil. Trans. R. Soc., A, **227**, 21-62 (1927).

[34] Terr. Mag., **37**, 42 (1932), for the years 1906 to 1930; **39**, 201-202 (1934), for the years 1923 to 1933 together with a similar diagram for sunspots.

to 21] of our series; the maximum of this wave falls near May 16. Amplitudes of less than $0.01C$ occur in rotations Nos. 61 and 372. The diagram will be referred to in later discussions.

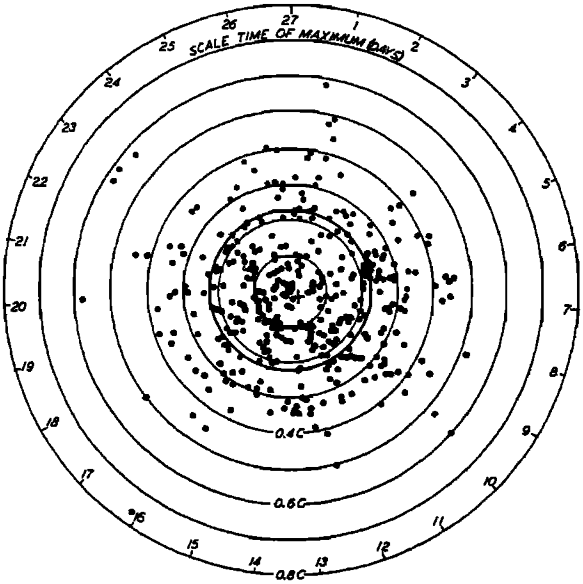


FIG. 2—HARMONIC DIAL, INTERNATIONAL MAGNETIC CHARACTER-FIGURE C, 1906-1933, FOR EACH OF 378 INTERVALS OF 27 DAYS BEGINNING JANUARY 11, 1906 (=DAY 1)—SINE-WAVES OF 27-DAY PERIOD

9. *Graphical interpretation of harmonic analysis*—While we shall not go into the much-discussed details of practical harmonic analysis,[35] that is, the actual evaluation of the equation (5.5) for the coefficients, a graphical interpretation of these equations, using the principle of superposition (section 5), will be helpful later.

The character-figures C for rotation No. 275, starting April 14, 1926, have been plotted in the top row of Figure 3*A*. This set of 27 ordinates can be conceived as a sum of 27 primitive sets, in each of which all ordinates are zero except one; the first three of these sets are plotted in Figure 3*A*. Generally speaking, the set of ordinates

$$y_1, y_2, y_3, \ldots, y_r \tag{9.1}$$

is equivalent to the sum of the primitive sets

$$\left\{\begin{array}{l} y_1,\ 0,\ 0, \ldots, 0 \\ 0,\ y_2,\ 0, \ldots, 0 \\ 0,\ 0,\ y_3, \ldots, 0 \\ \ldots\ldots\ldots\ldots\ldots \\ 0,\ 0,\ 0, \ldots, y_r \end{array}\right. \tag{9.2}$$

[35]For practical harmonic analysis see C. Runge and F. König, Numerisches Rechnen, pp. 208-231, Berlin, 1924; also E. T. Whittaker and G. Robinson, The calculus of observations, London, 1924. For some schemes used in geophysical applications see C. R. Duvall and C. C. Ennis, Terr. Mag., 32, 151-162 (1927); J. Bartels, Beitr. Geophysik, 28, 1-10 (1930); and the book of K. Stumpff already noted under footnote 10. A great help in numerical work is given by L. W. Pollak, Handweiser zur harmonischen Analyse (Prager Geophysikal. Studien Heft 2), which has appeared as Czechoslovakische Statistik, Reihe 12, Heft 10, Prague, 1928, while the same author's "Rechentafeln zur harmonischen Analyse," Leipzig, 1926, can in general be replaced by Crelle's Rechentafeln or by the slide-rule.

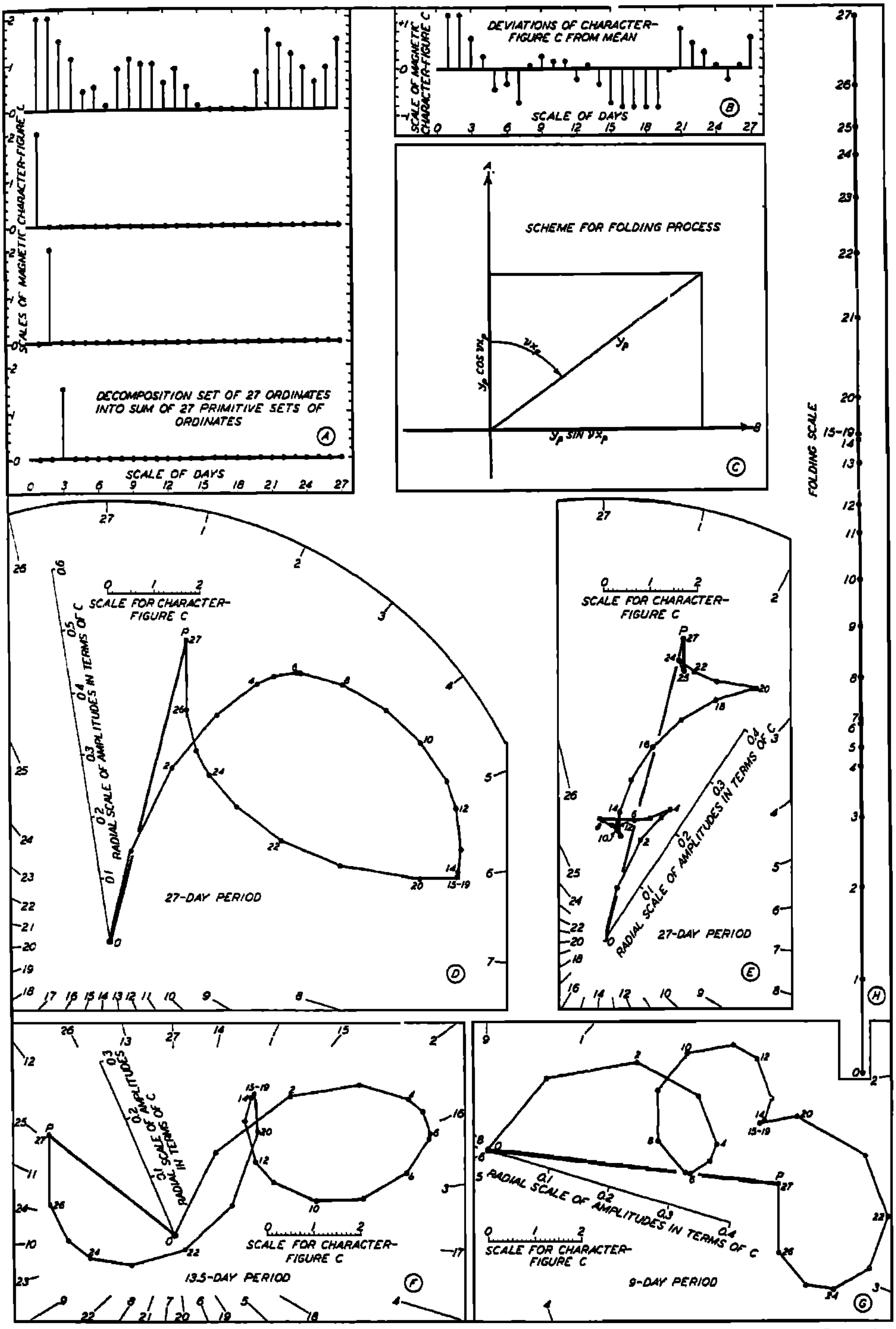


FIG. 3—GRAPHICAL HARMONIC ANALYSIS, OR FOLDING PROCESS, FOR INTERNATIONAL MAGNETIC CHARACTER-FIGURES FOR THE 27 DAYS, APRIL 14 TO MAY 10, 1926, FOR PERIODS OF 27, 13.5, AND 9 DAYS

According to (5.5), the harmonic coefficients, multiplied by $(r/2)$, for each primitive set are given, for the frequency ν, by $y_\rho \cos \nu x_\rho$, $y_\rho \sin \nu x_\rho$ (where $\rho = 1, 2, \ldots, r$): the representation in the harmonic dial (Fig. 3*C*) is a vector of length y_ρ forming the angle νx_ρ with the direction OA, because the projections of this vector are equal to the coefficients (times $r/2$). The sum of these r primitive vectors has, according to (5.5), the projections $(r/2)\, a_\nu$, $(r/2)\, b_\nu$ and represents therefore the sine-wave for the original set (9.1). In our example—the set of 27 ordinates in the top row of Figure 3*A*—$r = 27$, $x_1 = 2\pi/r = 13°.3$: for a sine-wave of 27-day period, $\nu = 1$, the angles νx_ρ for the successive primitive vectors are 13°.3, 26°.7, 40°.0, . . . , 360°.0, and the whole construction of summing the vectors consists in joining together the ordinates y_ρ, changing successively the direction clockwise by 13°.3 (Fig. 3*D*). The vector between O and the end-point, P, should be divided by $(r/2) = 27/2$ to obtain the amplitude c_1: instead, we can measure it in a scale enlarged (27/2) times (radial scale for OP indicated in Fig. 3*D*). Thus, we see from Figure 3*D*, comparing it with Figure 1, that the orthogonal coordinates of P, in units of C, are $a_1 = +0.48$, $b_1 = +0.13$, and its polar coordinates $c_1 = 0.50$, $\alpha_1 = 76°$: the sine-wave, therefore, is

$$+0.48 \cos x + 0.12 \sin x = 0.50 \sin (x + 76°) \tag{9.3}$$

Its maximum occurs about the time $x = 14°$, or $t = 14 \times (27/360) = 1.05$ days, or, since the time 1 day designates Greenwich noon of April 14, 1926, about 1 o'clock in the afternoon of that day.

If all the ordinates $y_1, y_2, \ldots y_r$ were equal, the construction in Figure 3*D* would lead to a regular polygon ending at the origin, that is, to vanishing coefficients, as could be expected. From the principle of superposition it follows, therefore, that a positive or negative constant can be added to all ordinates without changing the harmonic coefficients. For instance, the arithmetic mean a_0 can be subtracted, which amounts to measuring the ordinates in positive or negative *deviations* from the level a_0 (Fig. 3*B*): the construction of Figure 3*E*, plotting negative ordinates in the reverse direction, leads, of course, to the same point P as Figure 3*D*.

Figures 3*F* and 3*G* are analogous to Figure 3*D* and show the construction of the harmonic coefficients with frequencies $\nu = 2$ and 3, or periods of 13.5 and 9 days. The scales for the time of maximum are entered on scales around Figures 3*F* and 3*G*, while the scales for OP are the same in all diagrams 3*D*, 3*E*, 3*F*, and 3*G*. The sine-waves of frequencies 2 and 3, in units of C, are

(9.4) $+0.16 \cos 2x - 0.20 \sin 2x = 0.26 \sin (2x + 141°)$ with maxima on days 11.6 and 25.1

(9.5) $-0.06 \cos 3x + 0.46 \sin 3x = 0.46 \sin (3x + 353°)$ with maxima on days 2.4, 11.4, and 20.4

10. *The harmonic folding process*—The constructions in Figures 3*D*, 3*F*, and 3*G* can be interpreted as follows: Imagine a *folding scale* having links of the lengths of the ordinates y_ρ. If stretched out, as illustrated in Figure 3*H*, its entire length is equal to the sum of the ordinates, in the general case, $r\, a_0$. Suppose now each joint is turned by the angle $360°/27 = 13°.3$; we then obtain Figure 3*D*; by turning each

joint by $2\times360°/27$ or $3\times360°/27$, we obtain Figures $3F$ and $3G$. In general, this bending of the folding scale by the angles $\nu\times2\pi/r$ furnishes $(r/2)\ a_\nu$ and $(r/2)\ b_\nu$, and the distance of the end-point from the origin is $(r/2)\ c_\nu$. This idea of the ***harmonic folding process***, as it can be termed, will be helpful later.

Usually, only the result of the folding process, P, is retained in the harmonic dial. Nevertheless, and although for most actual computations numerical or mechanical harmonic analysis is preferable to graphical analysis, it is sometimes useful to recall the folding process as producing the vector OP, because it reveals the contribution of each single ordinate to the final vector. This contribution is particularly clear in the folding of the deviations from the arithmetic mean [the folding rule itself having then positive and negative links]; in Figure $3E$, the large positive ordinates 1 to 3 and the large negative ordinates 15 to 18 make the largest strides towards P. For illustration, Figure 4 shows, for an exact cosine-

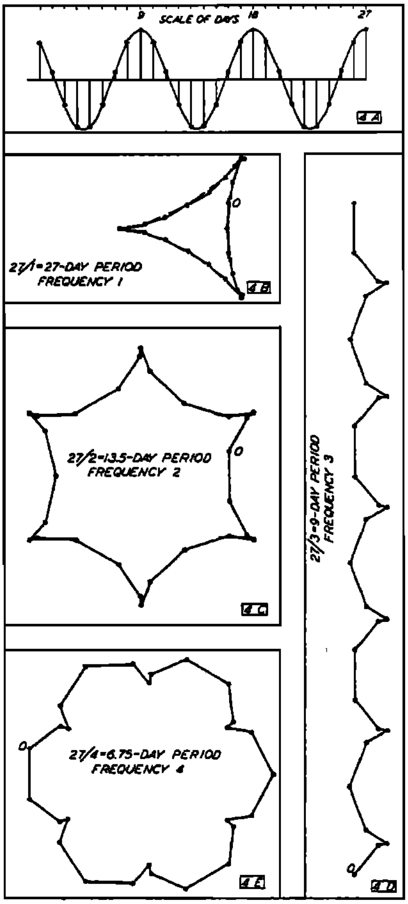


FIG. 4—A PERIOD OF FREQUENCY 3 IN 27 ORDINATES AND ITS GRAPHICAL HARMONIC ANALYSIS, OR FOLDING PROCESS, FOR PERIODS OF FREQUENCY 1,2,3,& 4

wave of 9-day periods, the folding process for the frequencies 1 to 4 (periods 27, 13.5, 9, and 6.75 days) of a set of 27 days.

The folding process is also a good illustration of the remark at the end of section 7.

11. *Standard deviations for sine-waves and residuals*—An *exact* representation of the set of r ordinates y_ρ is obtained by a series $\phi_k(x)$ of the form (5.2) if it is extended so that the number of coefficients $a_0, a_1, b_1, \ldots a_k, b_k$ is equal to that of the ordinates; if r is uneven, then $k=(r-1)/2$, while for even values of r the last term is $a_{(r/2)} \cos(rx/2)$, with $a_{(r/2)}$ given by (5.6). This procedure to represent a set of ordinates in the interval $t=0$ to T as a sum of sine-waves, as well as the approximation obtained for smaller values of k, is a *purely mathematical affair* and involves in no way the physical nature of the phenomenon described by these ordinates. Especially the fact that the sum $\phi_k(x)$ is periodic, repeating its values after intervals which are entire multiples of T, does not imply a similar property of the geophysical phenomenon outside the range of observation. The question of the physical meaning of the various sine-waves, and the possibility of "*forecasting*" *by means of periodicities* requires, therefore, additional tests, statistical in nature, which will be discussed later.

With less than r coefficients, the series $\phi_k(x)$ gives only an approximation, the degree of which can be estimated in the following way: The deviations of the given ordinates y_ρ from their respective arithmetic mean a_0 may be called

(11.1) $$z_\rho = y_\rho - a_0 \text{ (for values of } \rho = 0, 1, 2, \ldots r)$$

The standard deviation ζ may be defined as usual, that is, ζ^2 is the average of the z_ρ^2. It can be easily calculated from the y_ρ^2 and a_0: for $z_\rho^2 = y_\rho^2 - 2y_\rho a_0 + a_0^2$ and summing over $\rho=1$ to r gives $\Sigma z_\rho^2 = \Sigma y_\rho^2 - 2a_0\ \Sigma y_\rho + ra_0^2$; replacing Σy_ρ by ra_0 and dividing by r, we obtain the well-known formula

(11.2) $$\zeta^2 = \Sigma y_\rho^2/r - a_0^2$$

It can be shown (Appendix 1) that the average value of $\phi_k(x_\rho)$ is a_0, and its standard deviation η_k is given by $\eta_k^2 = (a_1^2+b_1^2+a_2^2+b_2^2+\ldots+a_k^2+b_k^2)/2$ or, applying (6.2)

(11.3) $$\eta_k^2 = (c_1^2+c_2^2+\ldots+c_k^2)/2$$

except in the [geophysically irrelevant] case of the exact representation and r even, when the last term in the bracket is $2\, a_{(r/2)}^2$. Furthermore, the standard deviation s_k of the residuals, defined by (5.3), yields, on evaluation (see Appendix 1), the remarkably simple expression

(11.4) $$s_k^2 = \zeta^2 - \eta_k^2$$

or

(11.5) $$s_k^2 = \zeta^2 - (c_1^2+c_2^2+\ldots+c_k^2)/2 = (1/r)\ \Sigma\ y_\rho^2 - a_0^2 - (c_1^2+c_2^2+\ldots+c_k^2)/2$$

Each additional harmonic term reduces, therefore, the residuals by subtracting half of its squared amplitude from ζ^2, the squared standard deviation of the given ordinates. This applies also if only one or a few terms of ϕ_k are selected, for instance, the waves with frequencies 2 and 4.

In the case of exact representation all residuals and therefore s_k^2 are zero so that, from (11.5),

$$(11.6)\qquad (c_1^2+\ldots+c_{l-1}^2)/2+a_l^2=\zeta^2 \text{ (with } r \text{ even, } l=r/2)$$

$$\text{or}\qquad (c_1^2+\ldots c_l^2)/2=\zeta^2 \text{ (with } r \text{ uneven, } l=(r-1)/2)$$

For convenience, we shall put, for r even, $a_{(r/2)}\sqrt{2}=c_{(r/2)}$, so that the second equation always holds. This equation (for convenience, we shall only consider the case of r uneven) furnishes an estimate for the *upper limit of the remaining coefficients*, if a number of coefficients, up to the index k, have already been computed; because, from (11.6) and (11.5)

$$(11.7)\qquad c_{k+1}^2+c_{k+2}^2+\ldots+c_l^2=2\,\zeta^2-c_1^2-c_2^2-\ldots-c_k^2=2\,s_k^2$$

The square of the largest coefficient among the coefficients of the terms with higher frequency than k can therefore be not larger than the right-hand side, $2\,s_k^2$.

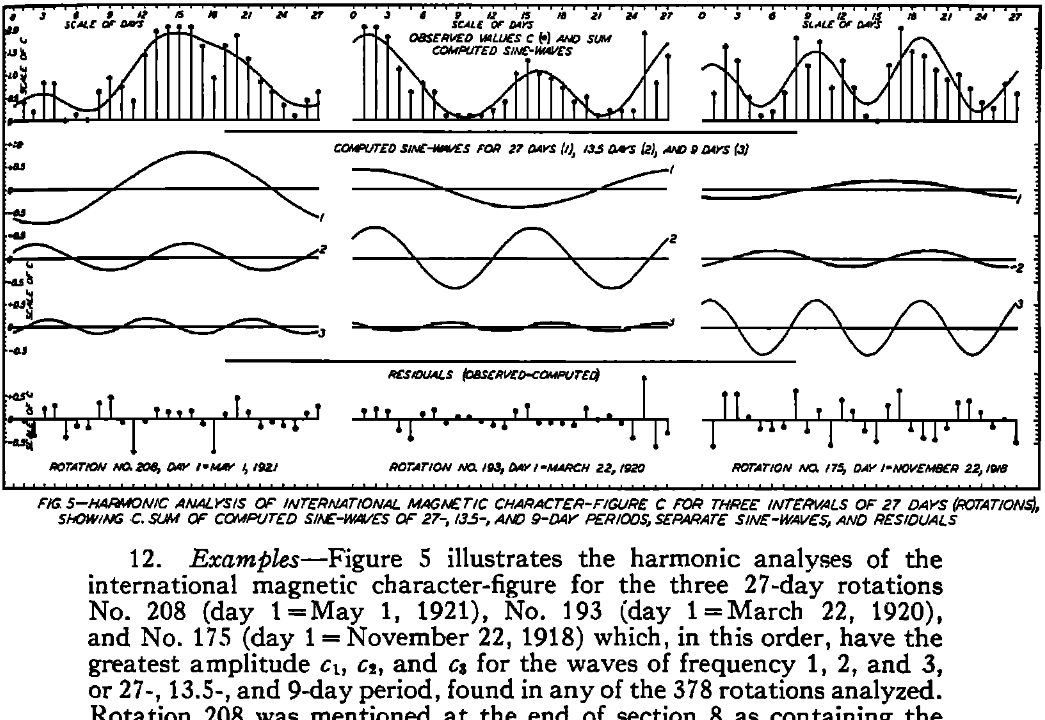


FIG. 5—HARMONIC ANALYSIS OF INTERNATIONAL MAGNETIC CHARACTER-FIGURE C FOR THREE INTERVALS OF 27 DAYS (ROTATIONS), SHOWING C, SUM OF COMPUTED SINE-WAVES OF 27-, 13.5-, AND 9-DAY PERIODS, SEPARATE SINE-WAVES, AND RESIDUALS

12. *Examples*—Figure 5 illustrates the harmonic analyses of the international magnetic character-figure for the three 27-day rotations No. 208 (day 1 = May 1, 1921), No. 193 (day 1 = March 22, 1920), and No. 175 (day 1 = November 22, 1918) which, in this order, have the greatest amplitude c_1, c_2, and c_3 for the waves of frequency 1, 2, and 3, or 27-, 13.5-, and 9-day period, found in any of the 378 rotations analyzed. Rotation 208 was mentioned at the end of section 8 as containing the heavy disturbances of May 12 to 21, 1921. Figure 5 gives, for each of the three rotations, in the first row the observed C, and the sum of three sine-waves, then the three sine-waves separately, and, finally, the residuals or differences between the observed C and the sum of the sine-waves. Some numerical values are given in Table 1; day 0 (or 27) is the origin of time, $\alpha=90°$ means that a maximum of the sine-wave occurs on day 27. The standard deviation ζ refers to the observed values of C, η_3 to the sum of the three sine-waves $[\eta_3^2=(c_1^2+c_2^2+c_3^2)/2]$, and s_3 to the residuals $(s_3^2=\zeta^2-\eta_3^2)$. The unit used is 0.01 unit of C.

Table 1—*Harmonic analysis of the three rotations with the largest amplitudes c_1, c_2, and c_3 (unit for a_0, c_1, c_2, c_3, ζ, η_3, and s_3 is 0.01C)*

Rotation number	Arithmetic mean	27-day period		13.5-day period		9-day period		Standard deviations		
	a_0	c_1	α_1	c_2	α_2	c_3	α_3	ζ	η_3	s_3
			°		°		°			
208	90	76	235	29	32	16	318	66	59	29
193	76	42	79	66	40	9	121	62	55	27
175	90	20	240	17	290	60	60	55	46	30

13. *Generalized harmonic dial*—The equations given in section 11 suggest the conception of a *generalized harmonic dial*[36] consisting of a rectangular coordinate-system in $2k$ dimensions, the axes assigned to $a_1, b_1, \ldots a_k, b_k$. Our set of r ordinates [or deviations z_ρ] is then represented by a single point P in this system, or the vector OP, and superposition is again represented by vector addition. The ordinary harmonic dials for the various frequencies are two-dimensional projections of the generalized dial. For the exact representation [if r is even, the last coordinate entered is not $a_{(r/2)}$, but $c_{(r/2)}=\sqrt{2}\times a_{(r/2)}$], the length of the vector OP is, according to (11.6), equal to $\zeta\sqrt{2}$. All sets of ordinates with the same standard deviation ζ are therefore exactly represented by a point on the sphere with radius $\zeta\sqrt{2}$. The formulae (11.4) and (11.5) for approximate representation can also be easily interpreted in this geometrical illustration. While, of course, actual drawings cannot be made, the conception of the generalized dial will be found useful in certain applications, especially for the transition from sine-waves to periodicities of other form (section 40).

14. *The ordinary periodogram*—The *periodogram* of a function f(t) in the interval $t=0$ to T is a diagram in which the amplitudes c_ν of the sine-waves are plotted against their frequencies ν or their periods T/ν. A. Schuster[8,9] himself favored later the use of c_ν^2 (instead of c_ν) as ordinate in order to simplify the statistical considerations based on the periodogram [or, as he called it, the periodograph], but for actual plotting c_ν is preferred as an illustration.

For the series of the international magnetic character-figure C for the years 1906-1926 used in Pollak's[11] paper, with a total of $r=7670$ days, an exact representation would be obtained by the same number of coefficients, namely, the average $a_0=0.62C$ [C is, as always, used to denote the unit of character-figures], 3834 amplitudes c_ν and as many phases α_ν, and, finally, the coefficient a_{3835} for a cosine-wave of two-day period. As in (11.6), we put again $c_{3835}=\sqrt{2}\ a_{3835}$. The periods p_ν of the successive waves of frequencies $\nu=1, 2, 3, \ldots$ would be, in days, $p_1=7670$, $p_2=3835$, $p_3=1917.5$, ..., $p_{20}=383.5$, $p_{21}=365.2$ (a year), ..., $p_{42}=182.6$ (6 months), ..., $p_{191}=40.16$, $p_{192}=39.95$, ..., $p_{255}=30.08$, $p_{256}=29.96$, ..., $p_{852}=9.002$, $p_{853}=8.992$, ..., $p_{958}=8.006$, ..., $p_{2556}=3.0008$, $p_{2557}=2.9996$, ..., $p_{3834}=2.0005$, $p_{3835}=2.0000$. These few values indicate that the difference in the length of the successive periods T/ν is very small for the high frequencies, because $p_\nu-p_{\nu+1}=$

[36] J. Bartels, Pub. Nat. Res. Council, Trans. Amer. Geophys. Union, 12th annual meeting, pp. 126-131 (1931).

$T/\nu(\nu+1)$ is approximately proportional to $1/\nu^2$. The labor involved in computing all coefficients would be very great. Pollak selected the periods 3 to 7 full days and 21 to 40 full days and their halves and thirds, because they are most easily calculated, the angles νx_p in (5.5) repeating themselves after each period.

The fact that most of these periods selected by Pollak are not entire submultiples of 7670 must not be overlooked; however, if we omit a few days at the end of the series, they become submultiples of the slightly reduced number of days [for instance, 17 goes in 7667], so that the amplitudes c can still be said to be derived from practically the whole series.

Around the periods 9.00 and 30.0 days, waves for the additional periods of 8.95, 8.96, . . . , 9.05 days and 29.7, 29.8, . . . , 30.2 days were inserted by Pollak, using Darwin's scheme of approximation [section 38]. I have computed, in addition, the amplitudes for four submultiples of a year (periods 3, 4, 6, and 12 months). Pollak's periodogram, with these additions, is reproduced in Figure 6; for clearness, the scale for abscissae is not uniform, spreading between 8.95 to 9.05 and 29.5 to 30.5 days, and changing as indicated.

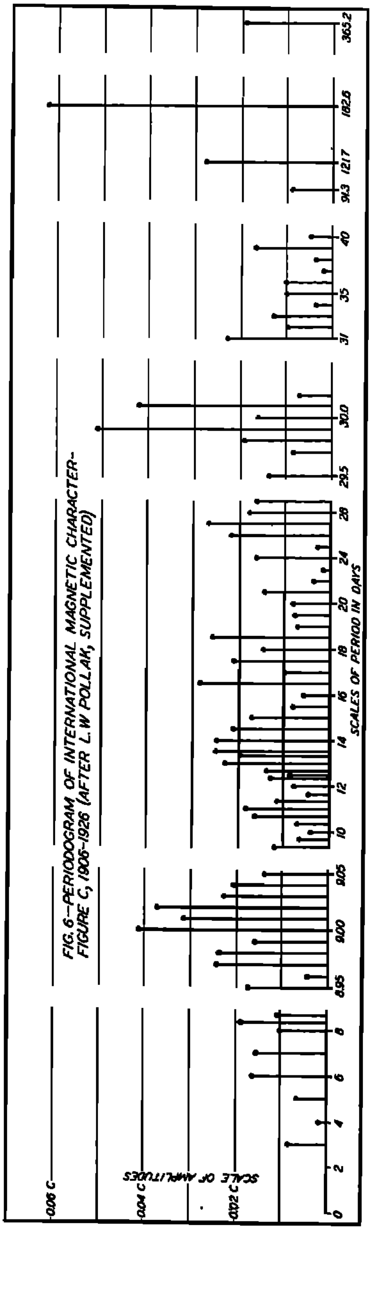


FIG. 6—PERIODOGRAM OF INTERNATIONAL MAGNETIC CHARACTER-FIGURE C, 1906–1926 (AFTER L. W POLLAK, SUPPLEMENTED)

The periodogram gives only the amplitudes c_ν. The phases a_ν could be indicated by writing them down, or making the periodogram three-dimensional, combining the separate harmonic dials for each frequency by aligning them along their origins, like wheels on a common axis, which would correspond to the base-line in the ordinary periodogram. A mixed two-dimensional dial of all frequencies, indicating c_ν, a_ν by vectors, seems, however, to be confusing.

15. *Discussion of Pollak's periodogram*—We shall apply formulae (11.3) to the 73 amplitudes c_ν calculated by Pollak. The sum of the c_ν^2 is $0.02388C^2$, and therefore the standard deviation η_k of the sum of these 73 sine-waves is given by $\eta_k^2 = 0.01194C^2$, or $\eta_k = 0.1093C$. This

means that the approximative series ϕ_k, which is the sum of the arithmetic mean $a_0=0.621C$ and of these 73 sine-waves, has values which deviate from a_0 only by a few tenths of the unit C. The poor approximation of ϕ_k is even better illustrated by applying (11.4). The standard deviation of the 7670 daily values of C, for the years 1906 to 1926, is $\zeta=0.461C$. The standard deviations of the residuals, s_k, is given by $s_k^2=\zeta^2-\eta_k^2=0.21266-0.01194=0.20072C^2$, or $s_k=0.448C$. If, therefore, the sum ϕ_k of the 73 sine-waves is subtracted from the given values of C, the fluctuation of the residuals, measured by $s_k=0.448C$, is practically the same as the fluctuation of the given values of C, measured by $\zeta=0.461C$.

Are, then, the 73 sine-waves of the selected frequencies at least distinguished by large amplitudes, as compared with the rest of the total of 3835 amplitudes? The answer is suggested by (11.7). The sum of c_ν^2 for the remaining 3762 sine-waves is $2s_k^2=0.40144C^2$. If all amplitudes except one were zero, this one amplitude would be $0.63C$—a case obviously ruled out by a mere glance at the original series. If, on the other hand, all remaining amplitudes should have the same value c', this would be given by $(c')^2=2s_k^2/3762$, or $c'=0.0103C$. The sum of the squares of 73 of the remaining amplitudes would then be $73(c')^2=0.0078C^2$. This is distinctively less than the sum of the squares of the 73 amplitudes for the actually selected waves, which above was given as $0.0239C^2$. The answer to our question is therefore affirmative.

We must remember, however, that the 73 selected frequencies are in no way equally distributed between all frequencies: from the list for the lengths of all periods given in section 14 it is seen that 958 periods are longer than 8 days, and the remaining 2877 shorter. Of the selected periods, 67 belong to the former and only six to the latter group: on the average, one out of 14 periods has been actually computed in the group of periods longer than eight days, but only one out of 480 periods in the group of shorter periods. This remark will be used later (section 32).

16. *Non-cyclic variation, selection-, or curvature-effect*—Harmonic analysis can be applied to all functions of time, $f(t)$, occurring in geophysics, and will result in a satisfactory approximation of $f(t)$ by a sum of sine-waves. It has already been said (section 11) that the significance of each sine-wave has to be tested, as will be described later. Apart from these tests, it will be easier to trace the real periodicities if such parts of $f(t)$, which are obviously non-periodic, are separated before the harmonic coefficients are discussed.

A typical case of a non-periodic part is the *secular variation* in terrestrial magnetism, which, in the course of a day or a month, can be considered as a linear one-sided trend. In computing diurnal variations, its effect is seen in a systematic difference between the values for successive midnights, the midnight-difference, or *non-cyclic variation*. Another, and even more effective, cause for non-cyclic variations in terrestrial magnetism is the recovery after disturbances. $f(t)$ can be freed from such effects by subtracting a suitable linear function of time, either by correcting the ordinates before the harmonic analysis, or by correcting the coefficients after the analysis. [The formulae deduced numerically by C. C. Ennis[37] can be derived in general terms (see ap-

[37]Terr. Mag., 32, 161-162 (1927).

pendix 4).] There has been some discussion on the feasibility of such *non-cyclic corrections*. They should be applied only if it is certain that the non-cyclic variation is due to an approximately linear function. This seems to be the case, for instance, in the average diurnal variations of magnetic horizontal intensity on quiet days, which show a systematic increase from midnight to midnight, and of those on disturbed days, showing a decrease.

More troublesome to eliminate is a systematic (mostly parabolic) *curvature*, which appears in *selecting* certain parts of a function of time. The classical case is the computation of the average diurnal variation of atmospheric pressure on clear and cloudy days in extra-tropical latitudes, which, in effect, amounts to selecting from the barogram and superposing, intervals of 24 hours between successive midnights, with high pressure (for clear days) and intervals with low pressure (for cloudy days). Now the general curvature of each single interval will be systematic so that, after non-cyclic correction, the average diurnal variation for clear days will show a pronounced maximum about noon, and that for cloudy days a pronounced minimum about noon. That these maxima and minima have nothing to do with an actual diurnal variation can be proved by selecting intervals of 24 hours between successive noons, which will show the maximum in the average clear-day variation about midnight. The possibility of such an effect, which was found in various phenomena by the author,[38] has often been overlooked, leading to curious misinterpretations. By suitable arrangement, this effect can be determined separately and corrected for.

(III) Statistical Principles—Random Walk

17. *The random walk with equal stretches*—The basis for all statistical considerations on periodicity is the problem of the "*random walk*," formulated, in its simplest case, by K. Pearson[39] as follows: "A man starts from a point O and walks a distance l in a straight line: he then turns through any angle whatever and walks a distance l in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $(r+dr)$ from his starting point, O." Figure 7[40] illustrates the case $n=27$: in addition, the random azimuths of the successive stretches are marked, in the upper left corner, by dots on a circle with radius l, in order to demonstrate (as in section 7) that the mass-center of these dots, as the average of the n-vectors of length l, is removed from the center of the circle by exactly $1/n$ of the distance between the starting and the end-point of the random walk.

The problem as well as its generalizations—for instance, to the case of stretches varying in length [section 18], or to more than two dimensions—has been amply discussed.[14-24] We need here only the following asymptotic expression for large values of n. Only the main theorems will be cited and discussed here; as to exact proofs, see the remark at the end of section 4.

[38] J. Bartels, Ann. Hydrogr., **51**, 153-160 (1923); Beitr. Physik frei. Atmos., **11**, 51-60 (1923); Terr. Mag., **37**, 18-20 (1932). See also S. Chapman and M. Austin, London, Quart. J. R. Met. Soc., **60**, 23-28 (1934).

[39] K. Pearson, Nature, **72**, 294 (1905).

[40] Constructed by taking the azimuth's from L. H. C. Tippett, Random sampling numbers (Tracts for computers, No. 15, Cambridge, 1927).

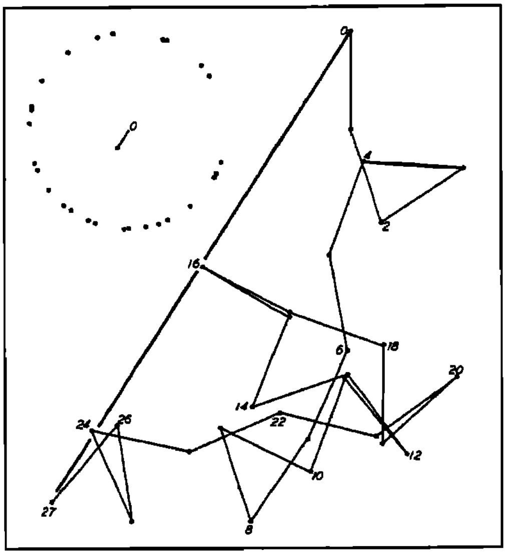


FIG. 7—RANDOM WALK WITH EQUAL STRETCHES

The random walk of n stretches may be repeated a great number (N) of times. The distance reached in each case may be called $L_1(n)$, $L_2(n)$, . . . , $L_N(n)$. Then it can be shown that the average square-distance, defined by

$$M^2(n) = \lim_{N\to\infty} [(L_1^2(n) + \ldots + L_N^2(n))/N] \tag{17.1}$$

is simply nl^2. $M(n)$, called the *expectancy*,[41] is therefore given by

$$M(n) = l\sqrt{n} \tag{17.2}$$

and the probability $w(r)\mathrm{d}r$ that a distance between r and $(r+\mathrm{d}r)$ is reached is (with $e^x = \exp x$) given by

$$w(r) = (2/M^2)\, r \exp(-r^2/M^2) \tag{17.3}$$

This curve, for which examples are given later (Fig. 9), reaches a maximum for $r = M/\sqrt{2}$ and has an inflection-point at $r = M\sqrt{6}/2$. As always, "probability" means distribution of "relative frequency," that is, $w(r)\mathrm{d}r$ is the limit, for $N\to\infty$, of the ratio of the number of distances falling between r and $(r+\mathrm{d}r)$, to the number N of all distances.

$w(r)\mathrm{d}r$ is the probability for the end-point falling between r and $(r+\mathrm{d}r)$, that is, within an area $2\pi r\mathrm{d}r$; the probability for the end-point falling within an infinitesimal area $\mathrm{d}a$ is therefore $(1/\pi M^2)\exp(-r^2/M^2)\mathrm{d}a$. If we plot these probabilities as vertical ordinates on the plane of the random walk, we obtain a symmetrical bell-shaped surface, produced by the rotation of a curve which is $(1/M\sqrt{\pi})$ times an ordinary normal Gaussian frequency-curve for standard deviation $M/\sqrt{2}$.

[41] This definition of the expectancy, as the square-root of the average square-distance, makes the formulae simple. Some authors prefer to call $M\sqrt{\pi}/2$ the expectancy for a reason given at the end of section 17.

It is convenient to express the distance r as a multiple of the expectancy M

(17.4) $$r = \kappa M$$

Then the probability that a distance between κM and $(\kappa + d\kappa)M$ is reached, is $w(\kappa)d\kappa$, with

(17.5) $$w(\kappa) = 2\kappa \exp(-\kappa^2)$$

The total probability $W(\kappa)$ that a distance greater than κM is reached, is obtained by integrating $w(\kappa)$ from κ to ∞, giving

(17.6) $$W(\kappa) = \exp(-\kappa^2)$$

TABLE 2—*Probability $W(\kappa) = \exp(-\kappa^2)$ that a random walk reaches a point beyond a circle with radius κM*

κ	$W(\kappa)$	κ	$W(\kappa)$	κ	$W(\kappa)$	κ	$W(\kappa)$
0.0000	1.0	0.8326	0.5	2.146	10^{-2}	4.015	10^{-7}
0.3246	0.9	0.9572	0.4	2.628	10^{-3}	4.292	10^{-8}
0.4724	0.8	1.097	0.3	3.035	10^{-4}	4.552	10^{-9}
0.5972	0.7	1.269	0.2	3.393	10^{-5}	4.799	10^{-10}
0.7147	0.6	1.517	0.1	3.717	10^{-6}	5.257	10^{-12}

The higher values in Table 2 apply, of course, only to large values of n, because, for instance, with $n = 16$, $M = 4l$, and the greatest possible distance, with all 16 stretches in line, is $16l = 4M$, so that $W(4) = 0$ in this case. For values of κ smaller than $\sqrt{n}$, however, the formula (17.6) is a very good approximation, and it is hardly ever necessary in geophysical applications to use the exact distribution-formulae worked out by K. Pearson,[16] and replacing (17.3) for small values of n; it is sufficient to note for later use that (17.2) remains valid for small values of n, including $n = 2$.

In Table 2, the value $\kappa = \sqrt{\log \text{nat } 2} = 0.8326$, with $W(0.8326) = 0.5$ is of special interest, because a circle with the radius 0.8326 M (usually called the *probable radius*, though this expression is misleading) divides the plane into two areas in which the end-point of the random walk may fall with equal probability.

The *arithmetic mean* of the $L_1, L_2, \ldots$, that is, limes $(L_1 + L_2 + \ldots + L_N)/N$, can be shown to be

(17.7) $$M\sqrt{\pi}/2 = 0.8862M$$

18. *Random walk with unequal stretches*—The statistics of a random walk, for which the successive stretches are unequal, say, $l_1, l_2, \ldots l_n$, obey, under certain conditions, the same set of formulae as that in section 17. The conditions and the proof are fully given by A. Khintchine[6]; it is sufficient here to say that the N sets of n stretches, $l'_1, l'_2, \ldots l'_n$; $l_1'', l_2'', \ldots l_n''$; $\ldots$; $l_1^{(N)}, l_2^{(N)}, \ldots l_n^{(N)}$, used for each walk must be taken at random from a common "supply" of stretches (nN in number); the

frequency-distribution of the lengths of the single stretches in this supply is arbitrary in wide limits[42]; for instance, it can be itself of the form of the equation (17.3). We obtain then the solution for the problem of the random walk if we simply define the *expectancy l of the single vectors* by

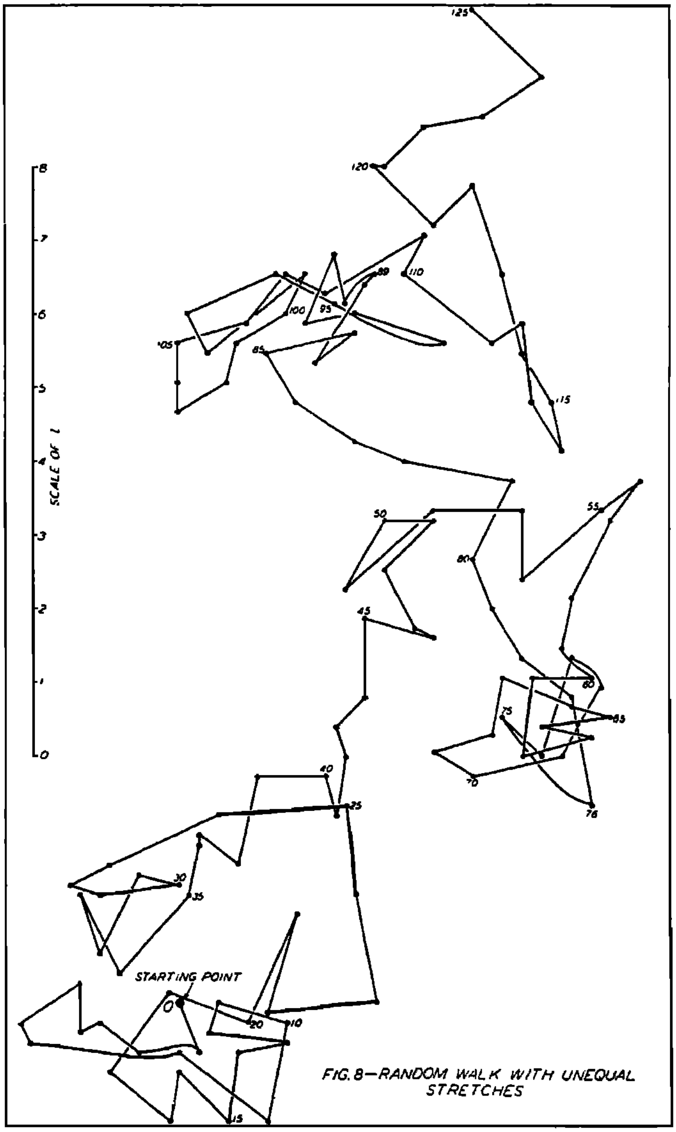


FIG. 8—RANDOM WALK WITH UNEQUAL STRETCHES

[42]This result was not known to K. Stumpff, who in his interesting paper on periodicities in sunspot-numbers (Prager Geophysikalische Studien, Heft 4—Čechoslovak. Statistik Reihe 12, Heft 14, Prague, 1930) discusses a special case of frequency-distribution, differing from the normal curve, and finds, of course, by rather intricate analysis, the general result of Khintchine for this special distribution.

(18.1) $$l^2=[(l_1')^2+\ldots+(l_n')^2+\ldots+(l_1^{(N)})^2+\ldots+(l_n^{(N)})^2]/nN$$

and apply the equations numbered (17.2) to (17.6); that is, the random walk leads, with the same probability, to distances between r and $(r+dr)$ from the origin as if it had consisted of n equal stretches of length l, where l is given by (18.1).

An example of such a random walk of $n=125$ stretches—again constructed with the help of Tippett's random numbers[40]—is given in Figure 8; the amplitudes are distributed around their average value l according to a normal Gaussian law with standard deviation $0.39l$ (derived from the random numbers in Sir Gilbert Walker's paper of 1930[27]).

19. *The expectancy for an average vector, the $1/\sqrt{n}$ law*—Our formulae can be readily used for another geometrical problem, which is only a formal modification of the random walk. If we conceive each stretch of length $l_1, l_2, \ldots l_n$ as a vector $\mathbf{l}_1, \mathbf{l}_2, \ldots \mathbf{l}_n$, the line between O and the end-point of the random walk is the vectorial sum $\mathbf{l}_1+\mathbf{l}_2+\ldots+\mathbf{l}_n$, and $1/n$ of its length is the average vector. If each single vector is plotted with O as starting point, and its end-point indicated by a dot, then the mass-center of the dots represents, again, the average vector $(\mathbf{l}_1+\mathbf{l}_2+\ldots+\mathbf{l}_n)/n$, much as indicated in section 7. The distribution of the average vector for a large number (N) of random walks (of n stretches each) is therefore a reduction of the distribution for the vectorial sum in the ratio $1:n$. For the sum, (17.2) gives the expectancy $l\sqrt{n}$; therefore the probability for the average vector is governed by (17.3) and (18.1), with the expectancy m (defined by the average square m^2 of the average vector) given by

(19.1) $$m=l/\sqrt{n}$$

Since l, according to (18.1), is the expectancy of the single vectors, and m that of the average of n vectors, we can formulate as follows: Averages for n random vectors have an expectancy which is the original expectancy of the single vectors reduced in the ratio $1/\sqrt{n}$.

20. *Comparison of harmonic dial and random walk*—The main application of the theory of probability to geophysics consists in finding, for a given set of observed quantities, a suitable statistical analogue which can be accepted as representing the idealized case reached if the number of observations, under the same conditions, could be infinitely increased.

The random walk, in the modification of section 19, offers itself as the statistical analogue to such harmonic dials as Figure 2, showing 378 sine-waves of 27-day period (amplitudes c, phases α) in the character-figure C. We shall first ask whether the "cloud" of 378 points on the dial is distributed so that each point can be regarded as the end-point of a random walk made under the same conditions. This puts $N=378$ and leaves n arbitrary. As the parameter governing the distribution we compute the expectancy M, where M^2, analogous to (17.1), is defined as the average of the squares, c^2, of the amplitudes for the 378 waves, and find $M=0.262C$.

In order to find the frequency-distribution, we count out how many amplitudes c fall in classes between equidistant limits. These limits, chosen according to the conventional rule that about 20 classes should be occupied, are 0, $0.036C$, $0.072C$, etc. The numbers of amplitudes in each class [the limits of which are marked by vertical lines] are entered as ordinates in Figure 9, and are compared with the theoretical frequency-distribution (probability), that is, with the curve computed, with the expectancy $M=0.262C$, from (17.6), the theoretical frequency between $\kappa_1 M$ and $\kappa_2 M$ being, of course, $N[W(\kappa_1)-W(\kappa_2)]$. The observed

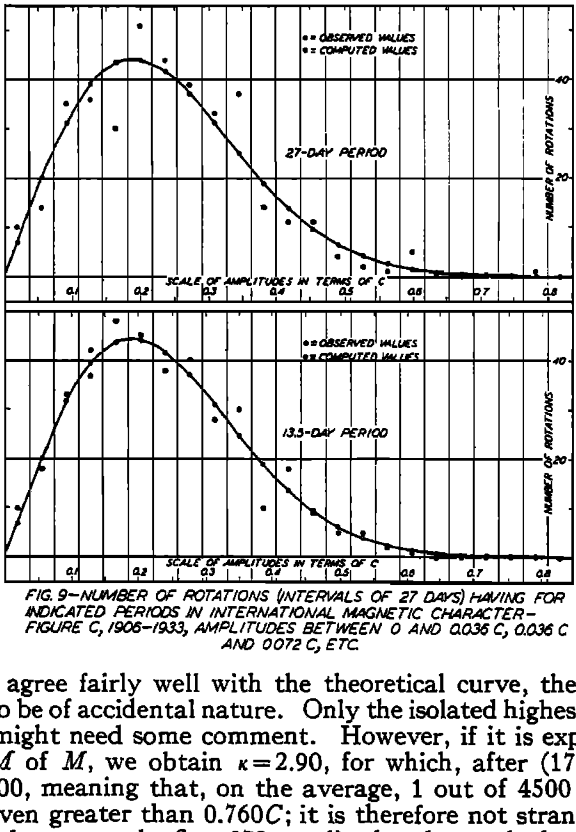


FIG. 9—NUMBER OF ROTATIONS (INTERVALS OF 27 DAYS) HAVING FOR INDICATED PERIODS IN INTERNATIONAL MAGNETIC CHARACTER-FIGURE C, 1906–1933, AMPLITUDES BETWEEN 0 AND 0.036 C, 0.036 C AND 0072 C, ETC.

frequencies agree fairly well with the theoretical curve, the differences appearing to be of accidental nature. Only the isolated highest amplitude $c=0.760C$ might need some comment. However, if it is expressed as a multiple κM of M, we obtain $\kappa=2.90$, for which, after (17.6) $W(\kappa)$ is about 1/4500, meaning that, on the average, 1 out of 4500 amplitudes should be even greater than $0.760C$; it is therefore not strange that one occurs already among the first 378 amplitudes observed, that is, in 1/12 of the average number 4500. [The limitation of C to values between 0.0 and 2.0 excludes, of course, amplitudes c of sine-waves over a certain theoretical limit, which implies a restriction on the use of (17.6) for higher values of κ. The theoretical limit mentioned for c is, by the way, not $1.00C$, as one might guess, but about $4/\pi=1.27C$, furnished, for instance, by a succession of 14 days with $C=2.0$ followed by 13 days with $C=0.0$.]

From the observations described in section 8, for each "rotation" of 27-day length, the sine-wave of frequency 2, or 13.5-day period, was also computed. The 378 amplitudes obtained in this way, applying the same analysis as in the case of the sine-waves with 27-day period, give the expectancy $M=0.264C$, and the frequency-curve in Figure 9. The highest amplitude is $0.657C$, with $\kappa=2.49$, and $W(2.49)\sim 1/500$, even greater than above, and practically not much different from 1/378.

Another test consists in deriving, from the 378 amplitudes, the "probable radius" and the "arithmetic mean," for the cloud, which should (section 17) theoretically equal $0.833M$ and $0.886M$, respectively. The cloud for the 27-day period actually gives for these ratios 0.86 and 0.88, and that for the 13.5-day period gives 0.82 and 0.87. The agreement with the theoretical values is satisfactory, because the deviations of these "observed" ratios from the theoretical values may be expected to be of the order $1/\sqrt{N}$, or 0.05. The probable radius is also drawn in Figure 2.

So far as these tests go—and *only* so far—each of the $N=378$ vectors in the harmonic dial can therefore be conceived as the result of a random walk of n stretches of lengths $l_1', l_2', \ldots l_n'; l_1'', l_2'', \ldots l_n''; \ldots l_1^{(378)}, \ldots l_n^{(378)}$, where the stretches vary at random around a mean-square value l, formed as in (18.1). Only the parameter $M=l\sqrt{n}$, the expectancy, is prescribed by the observations, while n can be chosen arbitrarily, with $l=M/\sqrt{n}$ following. Of course, the equivalent interpretation of section 19 can also be applied, conceiving each vector in the dial as the average vector of n random vectors.

21. *Harmonic dial and average vector*—This interpretation of section 19 can also be applied to our dial in Figure 2 in another way. This time we put $n=378$, and consider the hypothesis that our dial in Figure 2 is just a sample of a great number, N, of dials, each representing 378 vectors, with the same (or only slightly different) expectancy $l=0.262C$ for the single vectors. In each of these hypothetical dials, we consider the average vector, just as in Figure 2, where its end-point is indicated by a cross. Then, according to (19.1), the expectancy of this average vector is $m=l/\sqrt{n}=0.262/\sqrt{378}=0.0135C$, and the frequency-distribution around the origin is governed by (17.3), with the expectancy m put for M. Now, in our dial Figure 2, the average vector is actually found to be $0.0336C$, or $2.49m$. According to (17.6), a value exceeding $2.49m$ should occur only once in about 500 cases. Here it seems doubtful whether it might be assumed as merely accidental that, in the one and only dial actually obtained, a large average vector should be obtained such as might be expected only once in about 500 trials: still, the probability 1/500 for chance is generally considered not so small as to warrant a definite claim that the observations considered (in our case, the vectors plotted in Fig. 2) do not correspond to the statistical analogue (random walk) with which they are compared. By the way, 1/500 is roughly the chance, that, in throwing a coin, a predetermined side appears nine times in succession.

If, in all 378 sets, the 27-day period were perfectly persistent, that is, would have the same amplitude and the same phase (or time of maximum) persisting throughout the 28 years, then the average vector would be exactly equal to the single vectors, that is, the average amplitude would be M, instead of $m=M/\sqrt{378}$. This would give $\kappa=\sqrt{378}\backsim 20$, and the possibility $W(\kappa)$ for chance would become practically zero. If the 27-day period should vary at random, we would obtain values of κ around 1. The value $\kappa=2.49$ actually obtained could be interpreted as meaning that there is a probability of 500:1 that the 27-day period contains at least a small persistent part. We shall see later why this interpretation—which is commonly used in applications of periodogram-analysis—is not warranted (section 36).

In the case of the waves with 13.5-day period, we obtain the expectancy of the average vector $m=0.0136C$, while the average vector actually computed is $0.0200C=1.47m$, and the probability for chance is $W(1.47)=1/9$, much higher than in the case of the 27-day period.

22. *Remarks on the probability of chance* (*κ-test*)— We have followed the generally adopted convention in calling $W(\kappa)$ the "probability of chance." This is, of course, only a short expression for the exact definition of $W(\kappa)$, which may be repeated for the case of average vectors. From the amplitudes of n single vectors for the same frequency, we calculate the expectancy of these single amplitudes analogous to (18.1), divide it by $\sqrt{n}$ and thus obtain, according to (19.1), the expectancy for the amplitude of the average vector of that frequency; this expectancy, our m of sections 19 and 21, will be called $\mathfrak{c}$ from now on. It is based on the assumption of complete independence of the single vectors. By actual calculation (*vectorial* sum, division by n) of the average vector, we find its amplitude c, and calculate $\kappa=c/\mathfrak{c}$. $W(\kappa)$ is exactly the probability that, under random-walk conditions, an amplitude greater than $c=\kappa\mathfrak{c}$ should be found. In other words, if the random walk is repeated N times, about $NW(\kappa)$ times a distance greater than $\kappa\mathfrak{c}$ should be reached, or about once if it is repeated $(1/W(\kappa))$ times. If $W(\kappa)$ is very small, that is, $[(1/W(\kappa)]$ is very large, it is reasonable to assume that the conditions of random walk, or pure chance, do not hold because in the one and only case considered we have obtained a result which *should* occur only very rarely, and the suspicion is justified that some systematic regularity is contained in the distribution of the single vectors—which will be seen later.

Ad. Schmidt and Sir Gilbert Walker[27] have called attention to the following point: If only *one* frequency is considered, the considerations regarding $W(\kappa)$ hold. But some authors, having calculated $\mathfrak{c}$, c, and κ for each of a number (say 100) of independent frequencies, picked out that frequency with largest κ, say, κ_1. Then, of course, we must ask for the probability that once in 100 independent cases a value greater than κ_1 times its expectancy should occur, and that is 100 $W(\kappa_1)$.

Since the question whether $W(\kappa_1)$ is small enough in order to exclude chance is a matter of opinion anyway—one in a million is often considered as an upper limit—it is not necessary in most cases to consider the more accurate formulae introduced by Sir Gilbert Walker. He asks for the probability that, on random-walk conditions, the 100 independent values of κ should all be smaller than κ_1, and finds$\{1-[1-W(\kappa_1)]^{100}\}$; this, however, is, for small $W(\kappa_1)$, practically 100 $W(\kappa_1)$. If the observational material is large enough, some objections raised by Brunt[27] do not hold. Much more serious is a common mistake in the choice of the expectancy $\mathfrak{c}$, which it is the object of this paper to indicate (section 32 and following).

23. *Elliptical distributions*—The discussion of the properties of N sine-waves of the same frequency has, by the harmonic dial, been transformed into the geometrical analysis of the equivalent N vectors plotted from the origin, or of the "cloud" formed by their N end-points. We have seen that, under random-walk conditions, this cloud approaches

circular symmetry around the origin for large values of N. In actual geophysical work, however, especially for diurnal and seasonal variations, the cloud can have quite different shapes. If, for instance, the phenomenon contains a regular sine-wave of the frequency considered, with constant phase and amplitude, which is superposed by random fluctuations (introduced, for instance, by errors of observation), the cloud of points will be circular, but centered around the point A representing the regular sine-wave instead of around the origin O. If the regular sine-wave has constant phase, but a varying amplitude, the cloud is stretched into elliptical shape, and this elliptical distribution will be recognized most easily if the superposed irregular fluctuations are comparatively small.

Such elliptical distributions, which, in the most general case, have been discussed from the standpoint of the theory of probability by A. Khintchine,[6] have been found in the diurnal variations of terrestrial magnetism; statistical methods for computing the ellipses and further discussion of their physical meaning are given in a former paper in this JOURNAL.[2]

If the center A of the cloud is well outside the origin O, it is sometimes desirable to consider each single vector $\mathbf{c}=OP$ to consist of the regular vector $OA=\mathbf{r}$ (with constant amplitude r) and an irregular part $AP=\mathbf{i}$

$$\mathbf{i}=\mathbf{c}-\mathbf{r} \tag{23.1}$$

The expectancies of $\mathbf{c}$ and $\mathbf{i}$—computed in the usual way by summing the squares of the amplitudes, dividing by the number of vectors, and taking square root—may be c and i. The following relation is convenient for changing from c to i, or vice versa, namely

$$i^2=c^2-r^2 \tag{23.2}$$

This formula is a two-dimensional generalization of (11.2), because (23.1) corresponds to (11.1). [The proof is simple: The coefficients a and b of the vectors $\mathbf{i}$ and $\mathbf{c}$ follow (11.1) and (11.2) separately, and the squares of the vector-amplitudes are a^2+b^2.] If, therefore, we have calculated, for a cloud of points, the expectancy for distances of these points from any origin O, we obtain the expectancy for the distances of the points from their mass-center A by subtracting r^2, where r is the distance OA.

24. *The expectancy of sine-wave amplitudes calculated from rare events occurring at random*—In the preceding paragraphs, we have applied the conception of the random walk to vectors representing sine-waves in the harmonic dial. It can, however, also be applied to the actual calculation of the harmonic coefficients, as represented in the folding process (section 10), and this will establish a statistical relation between a set of random ordinates and its harmonic coefficients, which was the starting point of A. Schuster.[7]

The original conception of the random walk, with stretches of equal length, is the geometrical expression for the harmonic analysis of a function of the following type: Consider a long time-interval T, (20 years, say), divided into a large number r of equal intervals (about ten million minutes of time). The ordinates are put equal to 1 for minutes characterized by a (comparatively rare) event, for instance, the beginning of a magnetic storm with sudden commencement at a given observatory, and zero for all other minutes. The total number r_1 of events, or ordinates 1 will then be small (in our example not more than, say, 500) compared with the number r_0 of ordinates O ($r_1 << r_0$) and they will be scattered over the whole interval T considered. Take, then, a sine-wave of high frequency, say, $\kappa = 240$, with a period of one month; in the folding process, this means that the directions of the links describe a full swing of 360° per month. If, now, the events are scattered at random (like the atomic disintegrations in radioactive material), the folding process will lead to a diagram equivalent to a random walk with $n = r_1$ stretches.

In order to introduce the theory of probability, we must again hypothesize that our interval of observation T is a sample of a large number N of such time-intervals of length T with the same statistical properties, the average number of events in each interval being r_1. For a given period, the random walk of the folding process will lead to distances $L_1(r_1)$, ... $L_N(r_1)$, and their relative frequency, or the distribution of the end-points, will be governed by the formulae (17.2) and (17.3), with $M = \sqrt{r_1}$. Now, according to sections 9 and 10, the amplitudes c_ν of the sine-waves are obtained by dividing the distance L by half of the number $(r_0 + r_1)$ of ordinates. If, therefore, we define, analogous to (17.1), the *expectancy* $\mathbf{c}_\nu$ of the amplitude by

$$\mathbf{c}_\nu^2 = (c_{\nu 1}^2 + c_{\nu 2}^2 + \ldots + c_{\nu N}^2)/N \tag{24.1}$$

we obtain

$$\mathbf{c}_\nu = 2\sqrt{r_1}/(r_0 + r_1) \tag{24.2}$$

The remarkable feature of this result is that the expectancy $\mathbf{c}_\nu$ does not depend on the length of the period, or the *frequency* ν.

25. *Random walk and folding process, equipartition of the variance*—Some caution is necessary in applying the idea of the random walk to the ordinary case of equidistant ordinates, in which the directions in the folding process are limited to a few submultiples of 360°, such as in Figures 3*D*-3*G*. The theorem can be formulated most clearly if we use in the folding process, not the ordinates y_ρ themselves, but their deviations z_ρ from the mean a_0, $z_\rho = y_\rho - a_0$ ($\rho = 1, 2, \ldots r$), as illustrated in Figure 3*E*.

A great number N of sets of r ordinates may be given. The average of all Nr ordinates shall be zero, and the sum of their squares may be $Nr\zeta^2$, so that ζ is their standard deviation; nothing else will be assumed except that the ordinates are "random numbers," quite independent of each other.

Such sets can, for instance, be obtained by drawing ordinates at random from a great supply of ordinates having normal (Gaussian) frequency-distribution with standard deviation ζ and combining them to sets of r each. In each individual set, the arithmetic mean a_0^* of the r ordinates will not be zero, and the standard deviation ζ^* will not be exactly ζ, but, on the average for all N sets, according to well-known statistical laws,

$$(a_0^*)^2 = \zeta^2/r \text{ and } (\zeta^*)^2 = \zeta^2(r-1)/r \tag{25.1}$$

This example is, however, by no means the most general case for which the following theorem holds, because the frequency-distribution of the supply of ordinates may deviate from the normal law in wide limits.[43]

Each set of r ordinates is subjected to harmonic analysis yielding, for r uneven, $(r-1)/2$ amplitudes c_ν and phases α_ν; for r even we obtain $(r-2)/2$ amplitudes c_ν and phases α_ν and $a_{(r/2)}$. We form, for each frequency ν separately, the average square amplitude, or expectancy, $\mathbf{c}_\nu$ for all N sets, defined by (24.1) and, in the same way, $\mathbf{a}_{r/2}$. Then it can be shown that

$$\text{(25.2)} \qquad \mathbf{c}_\nu = 2\zeta/\sqrt{r};\ \mathbf{a}_{r/2} = \zeta/\sqrt{r}$$

The independence, for random ordinates, of $\mathbf{c}_\nu$ of frequency ν can be termed the law of the *equipartition of the variance* (where variance is the expression introduced by R. E. Fisher[44] for the square of the standard deviation). Because, taking the case of r uneven, we know from (11.3) that each amplitude c_ν contributes $c_\nu^2/2$ to the variance η_k^2 of the sum ϕ_k of sine-waves. If, therefore, we write down (11.3) for each set, sum up, and divide by N, we obtain

$$\text{(25.3)} \qquad (\zeta^*)^2 = (\mathbf{c}_1^2 + \mathbf{c}_2^2 + \ldots + \mathbf{c}_{(r-1)/2}^2)/2$$

If we assume[45] equipartition, or $\mathbf{c}_1^2 = \mathbf{c}_2^2 = \ldots = \mathbf{c}_{(r-1)/2}^2 = \mathbf{c}^2$, we obtain $(\zeta^*)^2 = \mathbf{c}^2(r-1)/4$. Remembering that, because of (25.1), $(\zeta^*)^2 = \zeta^2 (r-1)/r$, we obtain $\mathbf{c}^2 = \zeta^2\, 4/r$, that is, (25.2).[46]

The former formula (24.2) appears now as a special case of (25.2), because, in the example considered in section 24, $\zeta = \sqrt{r_1/r}$ and $r = r_0 + r_1$.

In a single set of r ordinates, c_ν can have any (positive) value, but the frequency-distribution of c_ν in a large number N of sets is governed by (17.3) to (17.6), with $M = \mathbf{c}$; the total probability that a single c_ν exceeds $M\mathbf{c}$, is again $W(\kappa) = \exp(-\kappa^2)$.

26. *Periodogram for random fluctuations*—The periodogram, as defined in section 14, can be plotted for each of the N sets of r ordinates considered in section 25; each periodogram shows, against the abscissae ν, the individual amplitudes c_ν as ordinates which, if it is desired, can be connected by a more or less arbitrary line. The *mean periodogram*, representing the average of all sets, shows *the expectancy* $\mathbf{c}_\nu$, defined by (24.1), as a function of ν; according to the law of equipartition (25.2) the mean periodogram for random ordinates would show a straight line at the distance $2\zeta/\sqrt{r}$ above the horizontal axis (only declining, for r even, to $\zeta/\sqrt{r}$ for the highest frequency $\nu = r/2$).

It may be noted that the mean periodogram does not only depend on the standard deviation ζ, but also on r. If, for instance, we divide 100,000 random ordinates into $N = 1000$ sets of $r = 100$ ordinates, the mean periodogram is only half as large as if we divide the material into $N = 4000$ sets of $r = 25$ ordinates.

The discussion of this paragraph applies at once to the case that the ordinates of any given function have accidental and independent *ob-*

[43] See section 18.

[44] R. E. Fisher, Statistical methods for research workers, 3rd ed., Edinburgh, 1930.

[45] The remarks given above are only illustrations, not a proof of the law of equipartition. For a simple proof, insert (5.5) into $c_\nu^2 = a_\nu^2 + b_\nu^2$ and add for all N sets.

[46] In the case considered, it has been necessary to distinguish between r and $(r-1)$, because r, the number of ordinates in a single set, may be as small as 3. But in all cases where the total number of observations appears in the equations, we shall generally not question scrupulously whether $(N-1)$ should stand for N, because we take N so large that this difference should not matter. In other words, observational material in which the addition or omission of one or a few observations should alter the conclusions seriously, is not considered sufficient for a statistical treatment.

servational errors, with standard error ζ. Then the periodogram of the true ordinates is superposed by the periodogram $2\zeta/\sqrt{r}$ of the errors, the superposition, for each separate frequency, following (23.2). The influence of observational errors on the harmonic coefficients is, however, mostly negligible in geophysical applications; it has been often mistaken for the influence of the actual irregular fluctuations of the observed quantity, which are fundamentally different in nature and will be shown to have, in each case, a peculiar type of mean periodogram (section 30).

(IV) PERSISTENT PERIODICITIES

27. *The expectancy as a function of the length of period*—The definition (24.1) of the expectancy $\mathbf{c}_\nu$ can, of course, at once be extended to the case that the Nr given ordinates represent a real geophysical phenomenon. The discussion in section 20 can therefore, simply by putting $\mathbf{c}$ for M, be expressed in the following way: With $N=378$ and $r=27$, that is, from 378 sets of 27 character-figures C for consecutive days (rotations), we obtain the expectancy $\mathbf{c}=0.262C$ for sine-waves of 27-day period computed from single rotations.

Since the expectancy $\mathbf{c}_\nu$ is the basis for all further discussion, it is necessary to consider the reliability of $\mathbf{c}_\nu$ if it is derived from N sets. It is clear that a single set, that is, a single amplitude c_ν, is a bad approximation for $\mathbf{c}_\nu$, because the single values of c_ν vary as expressed by (17.3) with $\mathbf{c}_\nu$ for M (see the probability-curves in Fig. 9). We imagine a very large supply of amplitudes c_ν. If we take, at random, N amplitudes c_ν from this supply, we shall compute an approximate expectancy $\mathbf{c}_\nu^{(N)}$ which differs from $\mathbf{c}_\nu$. If we repeat the computation for another set of N amplitudes, and another, etc., the $\mathbf{c}_\nu^{(N)}$ will be distributed around $\mathbf{c}_\nu$. This scattering, for large values of N over, say, at least 25, can be expressed by the standard deviation of the $\mathbf{c}_\nu^{(N)}$, which is approximately

$$\mathbf{c}_\nu/\sqrt{2N} \tag{27.1}$$

If N is large, this distribution around $\mathbf{c}_\nu$ approaches the normal law of errors; for smaller values of N, the distribution has been calculated by A. Schuster.[8]

We now turn to a fundamental consideration. The character-figures C for consecutive days are certainly not independent, since a magnetically quiet or disturbed time generally extends over a few days in succession. A number of statistical considerations are available for testing the degree of this dependence of consecutive values of C; for instance, by adding the figures C for two, three, and more consecutive days. If the standard deviation of C is $\zeta(1)$, and the standard deviations of the sums for 2, 3, etc., days, each divided by $\sqrt{2}$, $\sqrt{3}$, etc., are $\zeta(2)$, $\zeta(3)$, etc., respectively, then, on complete independence, we should expect $\zeta(1)=\zeta(2)=\zeta(3)=\ldots$, so that the ratios $\zeta(2)/\zeta(1)$, $\zeta(3)/\zeta(1)$, ... can be taken as measures of dependence. This test is mentioned here because its two-dimensional analogue will be used later for testing quasi-persistent periods.

Although, of course, no harmonic analysis is needed, and, in fact, would be clumsy for testing independence, we are, on the other hand, interested in the effect of dependence on the harmonic coefficients, especially, on the expectancy $\mathbf{c}_\nu$. It should, of course, make the expectancies for periods of a few days smaller than those for longer periods, instead of equipartition as expressed in (25.2). We shall test this assumption for the 27-day period in the series of international character-figures C. In this case, with $\zeta = 0.467C$ (this value of ζ for the interval 1906 to 1933 is only slightly higher than that, $0.461C$, for the interval 1906 to 1926 used in section 15) and $r = 27$, equipartition (obtained, for instance, by mixing up the daily figures at random) would give, from (25.2) the expectancy $2 \times 0.467/\sqrt{27} = 0.180C$. The actual expectancy $\mathbf{c}_\nu$ has been obtained for the 27-day period in section 20 (where it was called M), namely, $\mathbf{c}_\nu = 0.262C$, and its standard deviation, according to (27.1), is $0.262/\sqrt{2 \times 378} = 0.0095C$. The actual expectancy $0.262C$ exceeds therefore the equipartition value $0.180C$ by nearly nine times its standard deviation: the difference between the two values is therefore significant, not "accidental," and proves that the *expectancy* as derived by (24.1) from the actual amplitudes obtained by harmonic analysis from single rotations *depends definitely on the length of the period.*

28. *Persistent periodicities*—A sine-wave of period p is called *persistent* if it is repeated with the same amplitude and the same phase in all intervals of length p. Is it possible to trace such a persistent sine-wave if it is superposed on other fluctuations? The answer is affirmative, provided the number N of periods p contained in the interval of observations is sufficiently large. The procedure is suggested by the preceding discussion: Each single interval of length p is subjected to harmonic analysis and yields a sine-wave of period p, which, if represented in the harmonic dial of period p, is the vector-sum of two sine-waves, namely, the persistent sine-wave and another "accidental" sine-wave, for which the average square amplitude, calculated according to (24.1) from the N accidental sine-waves for the single intervals, may be $\mathbf{c}$. Then the average sine-wave of period p computed from all Np observations will be the vector-sum of the persistent wave (of amplitude c) and an average "accidental" sine-wave, the amplitude of which, according to section 19, is of the order $\mathbf{c}/\sqrt{N}$. Therefore, so small as c may be as compared with $\mathbf{c}$, in the average taken over a sufficient number N of periods the persistent wave will finally overwhelm the "accidental" waves produced by the non-persistent fluctuations which mask the hidden periodicity in the original data.

This process of reducing the average of the accidental wave is best visualized in the harmonic dial for the period p: The dial showing the sine-waves obtained from single intervals of length p will be a cloud of points widely scattered; but the cloud on the dial of the average sine-waves of period p obtained from a number of intervals of length Np will be reduced with respect to the end-point A of the persistent sine-wave vector, in the ratio $1/\sqrt{N}$, till, with N increasing infinitely, the whole cloud contracts into A.

The determination of the atmospheric tides of lunar origin has been so far the greatest "triumph" of this $1/\sqrt{N}$ law,[47] because, at extra-tropical

[47] S. Chapman, London, Mon. Not. R. Astron. Soc., 78, 635-638 (1918); see also foot-note 50 and the second and third references in foot-note 32, and J. Bartels, Quart. J. R. Met. Soc., 51, 173-176 (1926). Since then, the determination of the lunar semidiurnal variation of atmospheric temperature at Batavia, 1866-1928, with an amplitude of 0°.009 Centigrade, has added an even better example: S. Chapman, London, Proc. R. Soc., A, 137, 1-24 (1932).

stations, the expectancy c is about 0.30 mm mercury, for semidiurnal waves computed from 24 hourly values of atmospheric pressure, while c is only 0.01 mm, so that 900 days are needed to bring the accidental waves down to the level of the persistent wave and 100 years to reduce the accidental part to about $c/6$.

In a wider sense, also such periods can be called persistent (and traced in the same way), which have a constant period p, a phase fluctuating a few degrees around an average value, and a variable amplitude. Most diurnal and annual waves in meteorological or terrestrial-magnetic phenomena are of this nature. The reduction of the elliptical distributions discussed in section 23 follows the same $1/\sqrt{N}$ law, unless the averages are taken for systematically selected single intervals (section 16).

29. *Example: The semiannual persistent wave in terrestrial-magnetic activity; the summation-dial*—In our 28-year series of international magnetic character-figure C, only the period of six months can be definitely considered as persistent. In Figure 10, the harmonic dials have been plotted for sine-waves of six-month period, at the left computed from

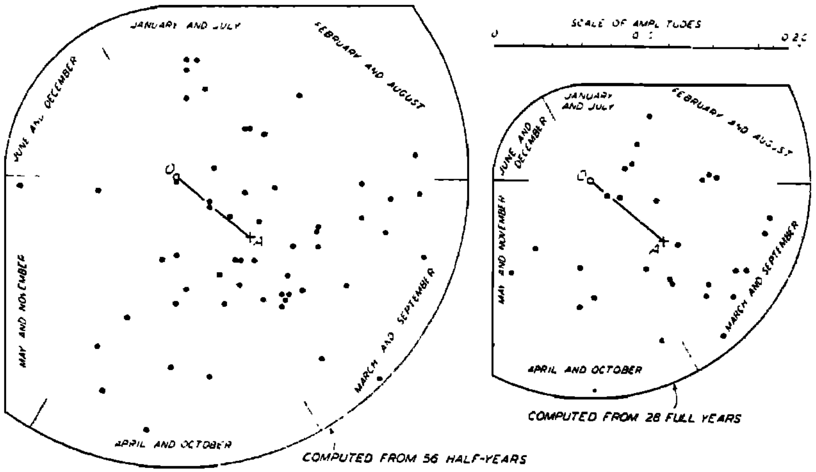


FIG 10—HARMONIC DIALS FOR 6-MONTHLY SINE-WAVES IN THE INTERNATIONAL MAGNETIC CHARACTER-FIGURE C

the 56 half-years, at the right computed from the 28 calendar years; of course (section 7), each dot in the right-hand diagram is the mass-center of two dots on the left. The average wave for all 28 years has the amplitude $c=0.0675C$, and its phase is given by maxima which occur about the dates March 22 and September 20, very near the equinoxes; it is represented by the average vector OA which, of course, is the same in both diagrams. The expectancy for single waves (vectors reckoned from origin O and combined according to (24.1)), is $c=0.111C$ at the left, $c=0.096C$ at the right. For the average of 56 or 28 accidental waves we should expect therefore $0.111/\sqrt{56}=0.0148C$, and $0.096/\sqrt{28}=0.0181C$. The actual average vector, $0.0675C$, is $\kappa=4.6$ and 3.6 times as large. With these values of κ, Table 2 for $W(\kappa)$ gives only a probability of about 10^{-9} or 10^{-6} that the waves are accidental. That the analysis based on the half-years gives even better results than that based on full years is easily understood, because the expectancy $0.096C$ obtained from full years is relatively more increased by the presence of the persistent wave than the expectancy, $0.111C$, obtained from half-years.

For the "accidental" or "irregular" vectors, reckoned from A, (23.2) gives the expectancy for single vectors $0.088C$ or $0.068C$, and for averages of 56 or 28 vectors, $0.0118C$ or $0.0129C$. This makes $\kappa = 5.7$ or 5.2, and $W(\kappa)$ smaller than 10^{-12}.

The basis of this discussion has been the comparison between a random walk and the gradual vectorial addition of the single vectors in the harmonic dial represented in Figure 10. This summation has been represented in Figure 11 (in a diagram which may be called *summation-dial*); the decisive preponderance of the directions indicating maxima near the equinoxes excludes all similarity with a "random walk" and illustrates the "reality" of the 6-month wave, which has just been quantitatively proven by the κ-test.

A more detailed analysis of this persistent semiannual wave and a discussion of its physical nature may be found in a former paper.[48]

30. *Mean periodogram for geophysical phenomena*—Our procedure of testing the reality of a periodicity consists in deriving a value for the expectancy c_ν which is based exclusively on harmonic analysis for single waves *of the same period*. This value represents therefore, in exactly the right manner, the combined effect of the standard deviation ζ of the given ordinates *and* the dependence of successive ordinates (section 27). The latter is present in most geophysical cases in so far as high values and low values of the ordinates occur in groups.[49] If, therefore, we cut the series of ordinates into sets of r successive ordinates, the arithmetic mean in each single set will, in general, differ *more* from the arithmetic mean of all ordinates than in the case of independence; in other words, the stand-

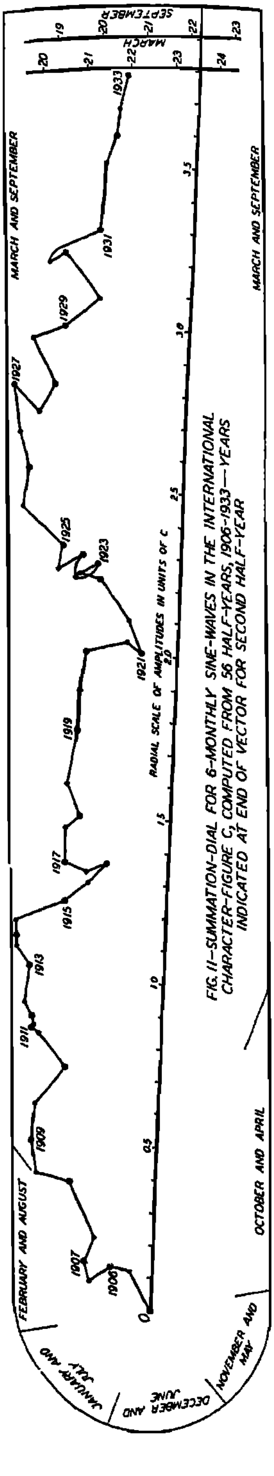


FIG. 11—SUMMATION-DIAL FOR 6-MONTHLY SINE-WAVES IN THE INTERNATIONAL CHARACTER-FIGURE C, COMPUTED FROM 56 HALF-YEARS, 1906-1933—YEARS INDICATED AT END OF VECTOR FOR SECOND HALF-YEAR

[48] Terr. Mag., 37, 22-27 (1932).

[49] The analogy to the Lexis theory of dispersion is obvious; this theory is described in every textbook on the theory of probability (for instance, that of Kamke[6] or R. E. Fisher[41]), and has been applied by F. Baur to meteorological phenomena [Met. Zs., 47, 381-389 (1930)]. In relation to periodicities, the Lexis theory must be modified, or specialized, as will be seen later (section 40).

ard deviation of the arithmetic means for sets of r successive ordinates will be *greater* than the random value $\zeta/\sqrt{r}$. [Example: International magnetic character-figure C, 1906 to 1933, has standard deviation for single daily values $\zeta = 0.467C$; if we form arithmetic means for each of the 378 rotations (intervals of $r = 27$ days), their standard deviation is found to be $0.148C$; if the values C for successive days were independent, this value should be only $0.467/\sqrt{27} = 0.090C(\pm 0.003)$.] On the other hand, if, in each single set, the deviations of each of the r ordinates from the arithmetic mean for that set are formed, their standard deviation ζ_r will be *smaller* than the standard deviation ζ of all ordinates, the ratio ζ_r/ζ increasing to unity with increasing r (in the case considered, $\zeta_{27} = 0.444C$). From (11.6), it follows therefore that the expectancy for smaller periods (computed from sets of a few ordinates) will be, in general, smaller than that for longer periods.

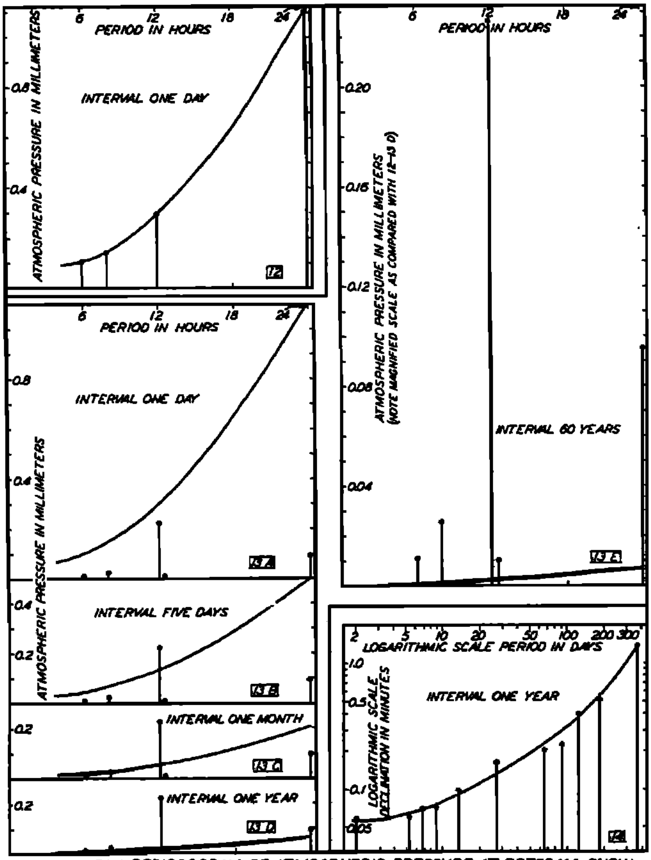


FIG. 12—MEAN PERIODOGRAM OF ATMOSPHERIC PRESSURE AT POTSDAM SHOWING MEAN AMPLITUDES OF SINE-WAVES BASED ON HARMONIC ANALYSIS FROM SINGLE-DAY INTERVALS FOR PERIODS 6, 8, 12, AND 24 SOLAR HOURS

FIG. 13—AMPLITUDES OF PERSISTENT SINE-WAVES FOR PERIODS 6, 8, 12, AND 24 SOLAR HOURS AND FOR 12 LUNAR (=12.4 SOLAR) HOURS CONTRASTED WITH MEAN PERIODOGRAMS FOR INTERVALS AS INDICATED

FIG. 14—MEAN PERIODOGRAM OF MAGNETIC DECLINATION AT GREENWICH, SHOWING MEAN AMPLITUDES OF SINE-WAVES BASED ON HARMONIC ANALYSIS FROM SINGLE-YEAR INTERVALS (AFTER SCHUSTER)

As an example, consider the hourly values of atmospheric pressure observed at Potsdam, Germany.[50] The expectancies for the sine-waves with 6-, 8-, 12-, and 24-hour periods, computed from single sets of $r=24$ hourly values, are, in mm of mercury, 0.11, 0.14, 0.30, and 1.11. They are entered in the mean periodogram of Figure 12. The free-hand curve drawn through the four ordinates can reasonably be expected to represent the actual mean periodogram, that is, the expectancy c as a function of the period p.

We now make the assumption—to be tested by its consequences—that the harmonic coefficients of period p, computed from different sets of r observations, are independent and do not contain a persistent part. Then the "random-walk" theory (19.1) is applicable, and we can at once obtain the mean periodogram for the amplitudes computed from the average harmonic coefficients won by harmonic analysis of sets of $2r$, $3r$, . . . , in general, of qr ordinates, because (7.1) the vectors for the average sine-waves derived by the harmonic analysis of qr ordinates are the averages of the q sine-waves computed from r ordinates each. From (19.1) it follows, therefore, that we obtain the mean periodogram for sets of qr ordinates simply by reducing the mean periodogram for sets of r ordinates in the ratio $1/\sqrt{q}$. This applies, of course, only to such periods p which are submultiples of the interval represented by r ordinates. The reduction can, by increasing q, in actual computation, be continued till the whole set, $q=Nr$, of available ordinates is subjected to harmonic analysis. Persistent waves of amplitudes c greater than $c/\sqrt{N}$, where c is the expectancy for that particular period, will then be discovered, and the ratio $\kappa=c/(c/\sqrt{N})$ will indicate the degree of reliability.

Figure 13 shows, in the curves, the mean periodogram for waves from 6- to 24-hour periods in atmospheric pressure at Potsdam, calculated from single days ($r=24$), and the mean periodogram for waves computed from $q=5$, 30, 365 days and 22,000 days (60 years), obtained by reducing the curve for single days in the ratio $1/\sqrt{q}$; for clearness, Figure 13*E* has a scale magnified ten times. The persistent waves of 6, 8, 12, and 24 solar hours, of amplitudes 0.011, 0.026, 0.226, and 0.095 mm have been indicated by vertical lines in each periodogram, and, in addition, the lunar tidal wave of period 12 hours 25 minutes, with amplitude 0.011 mm. It is striking how the persistent waves, with increasing number q of days, gradually pierce the mean periodogram, which represents the veil of the non-periodic fluctuations hiding the persistent waves. One year of observations (Fig. 13*D*) is sufficient to extract the solar 12- and 24-hourly waves, while 60 years of observations (Fig. 13*E*) are necessary to press the level of the mean periodogram in the neighborhood of 12-hour period down to one-fifth of the amplitude of the actual lunar tidal wave.

31. *A. Schuster's example of a mean periodogram*—The idea of the mean periodogram as drawn in Figure 12 is the outstanding contribution of A. Schuster to the study of periodicities. In fact, on page 122 of his second paper,[8] he gives an actual mean periodogram showing the same

[50] J. Bartels, Ueber die atmosphärischen Gezeiten, Berlin, Veröff. Preuss. Met. Inst., No. 346 (1927). The values given above are the *probable* radii, as they were actually calculated; the expectancies should be about 20 per cent higher, but that does not matter for our purpose here.

characteristic feature as our Figure 12. In order to show the details more clearly, logarithmic scales for both periods and amplitudes have been used in Figure 14, which represents Schuster's calculations based on the daily means of magnetic declination at Greenwich, 1871-1895, corrected for the non-cyclic variation due to secular variation (section 16), and shows the expectancies for sine-waves from 2- to 365-day periods supposed to be calculated from single years of observations, that is, the average amplitudes, computed according to (24.1) from the amplitudes obtained by harmonic analysis of $N=25$ sets of $r=365$ ordinates each. Figure 14 corresponds to Figure 12; the vertical lines give the expectancies as calculated, and the smooth line has been drawn to fit approximately.

A. Schuster used his values for the expectancy to test the presence of a persistent wave with period between 25.5 and 27.5 days. Since the whole interval of observation is 9160 days, these periods would represent the frequencies $9160/25.5=358$ and $9160/27.5=332$, so that about 26 independent sine-waves lie between these limits. The greatest among them, calculated from all $N=25$ years, has an amplitude of $c=0'.0785$, while the chance value, with $c=0'.163$ for a single year, is $c/\sqrt{N}=0.163/5=0'.033$. Therefore $\kappa=0.0785/0.033=2.4$. This value, according to (17.6), should, if pure chance were working, be exceeded once in about 300 cases, and it cannot be claimed to be unusual if an event, occurring, on the average, once in 300 cases, occurs already in the 26 cases actually considered. Schuster considers also periods which are not entire submultiples of 9160 (see section 38), which increase the number of "independent" periods between 25.5 and 27.5 days to 4 times our number 26, or about 100; this is even more unfavorable for a claim that the greatest period found indicates a persistent wave, because the probability for chance becomes as high as $100/300=1/3$.

32. *Erroneous applications of the periodogram*—Unfortunately, neither the original and powerful method of Schuster, just described, nor its equivalent in the harmonic dial, as developed since 1922 by the present author, have been applied in any of the later papers dealing with the periodogram. This seems to be due partly to an exaggerated conception regarding the amount of labor needed to compute a large number of harmonic coefficients, partly to the fact that Schuster himself, in his paper on sunspots,[9] does not use his own method.

Most of the recent papers on periodograms (for instance, those of Pollak[11]) and Stumpff[10]) use the following substitute for the exact methods: The harmonic coefficients for a number of selected periods [in Pollak's case (section 14), 73 periods ranging between 2 and 40 days] are computed from the whole observational material, without effective subdivisions (that means, r is taken as the number of all observations). The amplitudes for these "trial periods," or, in some cases, their squares, are summed and divided by the number of the trial-periods (in Pollak's case, 73); with this "expectancy," which is the substitute for our $c/\sqrt{N}$ of section 30, the amplitudes of the trial periods are compared, and the ratio κ of each amplitude to the "expectancy" is used to decide, by means of (17.6), on the reality of the large amplitudes.

A. Schuster,[9] in his paper on sunspots, recommends the following procedure for deducing the expectancy: From the whole of the observational material, without subdivisions, he computes the harmonic coefficients for a number of periods with lengths between 55 days and 24 years and enters their amplitudes in a periodogram. Then he goes on to say:

"It has been stated that in the absence of definite periods the expectancy of the intensity of the periodogram must be obtained from the periodogram itself in all cases where the events to be analyzed are not, as regards their succession, independent of each other. *The expectancy not depending on the period* we may select for the purpose any portion of the curve in which we have no reason to suspect any periodicities. The portion most suitable for this purpose in our case is that lying between 54 days and 1.5 years. Shorter periods must be avoided . . . owing to the fact that sunspots as a rule last several days. . . . Spots persist during more than one solar rotation. This effect will, however, disappear when the period is well above that of the solar rotation. When the periods come near to 1.5 years, the sub-periods of well-ascertained periodicities make their presence felt. Hence the limits chosen for calculating the natural intensity of the periodogram must be confined to about 35 days on the one hand and 1.5 years on the other."

It seems strange that Schuster, in the phrase printed here in italics, renounces his own discovery made in the second paper, and represented here in Figure 14. In fact, it is quite clear that in the case of sunspots the *expectancy must depend very largely on the period*, because of the general reasons discussed in section 30. This is confirmed by an independent calculation[48] of the expectancies for 6-monthly and 12-monthly periods in relative sunspot-numbers, 1872-1930; the amplitudes, calculated from single years, have, in the units of the relative sunspot-numbers, the expectancies 8.3 and 10.9. This distinct increase of the expectancy by about one-third of its value if the period lengthens from 6 to 12 months is likely to continue for longer periods.[51]

Now it seems extremely desirable to "clean the slate" of all uncertain periodicities and regard persistent periods as established only after the severest test. From this standpoint, the danger lies, of course, not so much in cases where the *assumed* expectancy for a certain period is greater than its proper value—though this might occasionally prevent the detection of an actual persistent wave—but in cases where it is *smaller*, because that makes the actually calculated amplitude appear more significant and entails higher values of κ. If, for instance, the proper value of κ is 2.15, indicating a probability for chance $W(\kappa)=1/100$, an underestimated expectancy assumed at half the proper value would yield $\kappa=4.3$, with $W(\kappa)=10^{-8}$, which would erroneously appear to justify a claim for a persistent wave. Table 2 (section 17) illustrates the serious mistakes possible if the expectancy is assumed too low, and, consequently, κ too high, even by as little as one-fourth of the proper value. And such an underestimate of the expectancy is almost certain if, as sometimes suggested, the largest amplitudes of the periodogram are omitted in calculating (in the manner indicated) the expectancy on the ground that they might indicate persistent waves and raise the expectancy unduly.

The periodogram has been discussed here because it has been used so often in previous work. The author prefers the illustration of persistent waves in the harmonic dial for their period, with the cloud of points contracting, with increasing number N of periods combined, into the end-point of the persistent vector (section 28). The harmonic dial

[51] K. Stumpff in his paper on periodicities in sunspots (see foot-note 42) distinguishes at least between short and long periods, the division being taken at about 3 years' length. He follows, however, Schuster in adopting a common expectancy for the longer periods.

confines the attention to the period for which the persistence is to be tested, and avoids the confusion produced by mixing amplitudes for periods of different length, and, therefore, of different expectancy.

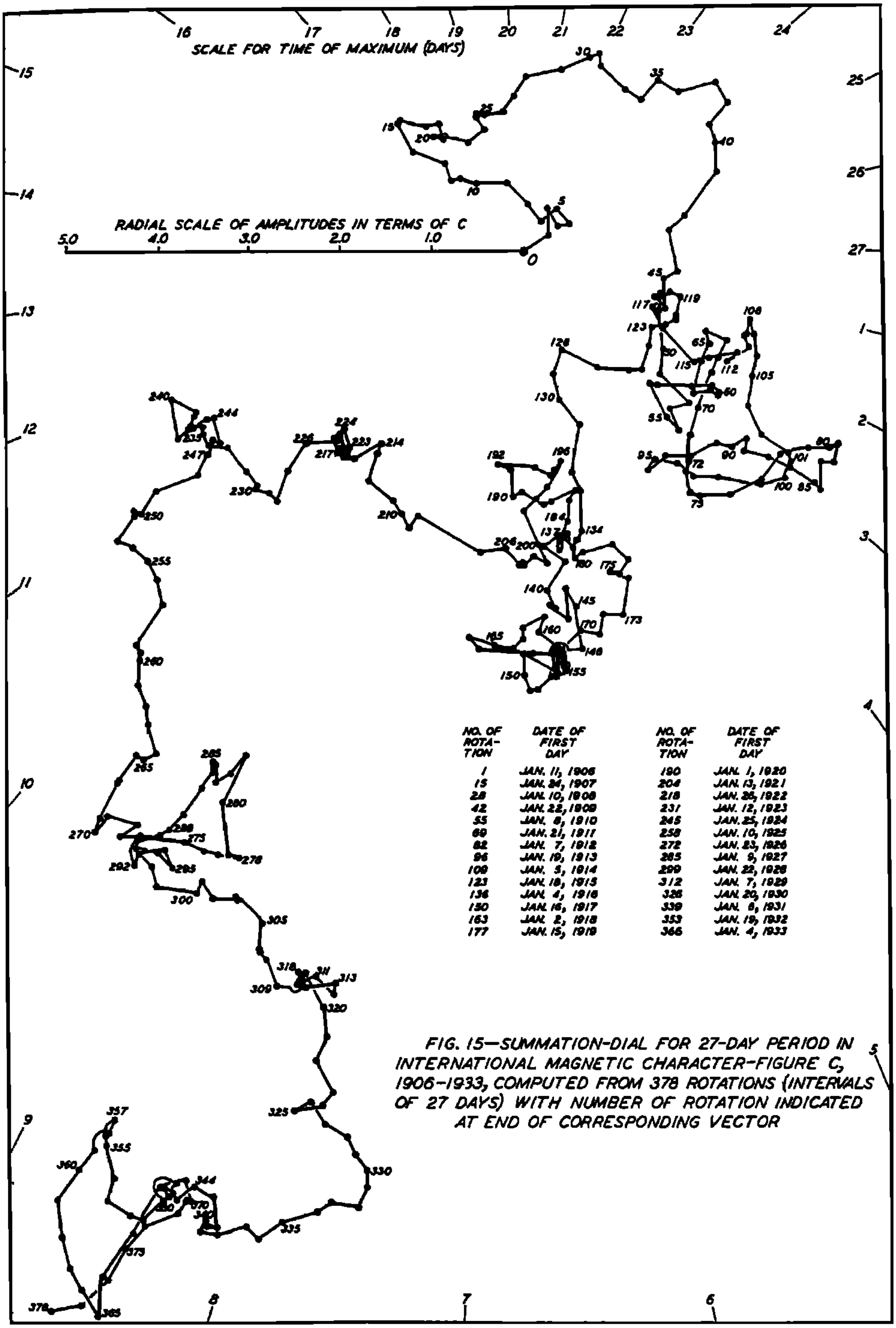
(V) QUASI-PERSISTENCE—EFFECTIVE EXPECTANCY

33. *Quasi-persistent waves*—We call quasi-persistent such periodicities which are repeated with approximately the same phase and amplitude for a certain number of periods, forming what may be termed a *sequence*, each sequence ending more or less abruptly without any relation to other sequences. This conception is not restricted to sine-waves; in fact, the most striking example is offered by the diagrams[34] for the 27-day recurrences in terrestrial-magnetic activity as described by the international magnetic character-figure C. This recurrence-phenomenon is expressed in quasi-persistence of the various sine-waves with periods that are submultiples of 27 days. Here we shall consider those with periods of 27, 13.5, and 9 days; later (section 40) we shall formulate our results without reference to harmonic analysis or sine-waves.

Quasi-persistence is best studied in connection with the summation-dial, introduced in section 29; summation-dials for the periods of 27 and 13.5 days are reproduced in Figures 15 and 16.[52] (In order to get a better reproduction, the dial in Figure 15 has been turned by 90° from that in Figure 2, vectors with maxima on day 27 pointing to the right.) These diagrams illustrate the vector-addition, step by step, of the single vectors in the harmonic dial for successive rotations; for instance (Fig. 15 for 27-day periods), the vector from the origin O to the point marked 130 in the summation-dial is the sum of all vectors for the single rotations 1 to 130, inclusive; a reduction to 1/130 would give the average vector for the interval of time covered by these 130 rotations. Of course, the summation-dial can also be used to form other averages than those starting at the origin. For instance, the vector connecting the points marked 130 and 378, divided by $(378-130)=248$ would be the average vector for rotations 131 (because the vector connecting points marked 130 and 131 refers to rotation 131) to 378, inclusive. Where the track returns to approximately the same point, the average vector for the intervening rotations is small; for instance, on the summation-dial for the 13.5-day period, the points 140 and 344 fall so close together that they are less than $0.05C$ apart, according to the scale for the single vectors. This means that the average vector (or amplitude for the 13.5-day period) for all the 204 rotations 141 to 344 comprising the whole interval between May 18, 1916, and June 16, 1931, is smaller than $0.05C/204$, or $0.00025C$.

On the other hand, we can select, on the same diagram, Figure 16, long distances traversed in a few rotations. For instance, the points marked 324 and 349 are $6.38C$ apart, indicating an average vector, for the 25 rotations Nos. 325 to 349, of $6.38/25=0.255C$. Applying considerations analogous to section 21 we find, for these 25 rotations, the expectancy for the single vector equal to $0.332C$, and the expectancy for the average of 25 vectors therefore $0.332/\sqrt{25}=0.0664C$; this gives $\kappa=0.255/0.0664=3.84$, $W(\kappa)=4\times10^{-7}$. The probability $W(\kappa)$ for

[52] Figure 15 represents the summation of the single vectors in Figure 2, while Figure 16 represents the same process for the 13.5-day period, for which the analogue to Figure 2 is not reproduced here.



NO. OF ROTA-TION	DATE OF FIRST DAY	NO. OF ROTA-TION	DATE OF FIRST DAY
1	JAN. 11, 1906	190	JAN. 1, 1920
15	JAN. 24, 1907	204	JAN. 13, 1921
28	JAN. 10, 1908	218	JAN. 26, 1922
42	JAN. 22, 1909	231	JAN. 12, 1923
55	JAN. 8, 1910	245	JAN. 25, 1924
69	JAN. 21, 1911	258	JAN. 10, 1925
82	JAN. 7, 1912	272	JAN. 23, 1926
96	JAN. 19, 1913	285	JAN. 9, 1927
109	JAN. 5, 1914	299	JAN. 22, 1928
123	JAN. 18, 1915	312	JAN. 7, 1929
136	JAN. 4, 1916	326	JAN. 20, 1930
150	JAN. 16, 1917	339	JAN. 6, 1931
163	JAN. 2, 1918	353	JAN. 19, 1932
177	JAN. 15, 1919	366	JAN. 4, 1933

FIG. 15—SUMMATION-DIAL FOR 27-DAY PERIOD IN INTERNATIONAL MAGNETIC CHARACTER-FIGURE C, 1906-1933, COMPUTED FROM 378 ROTATIONS (INTERVALS OF 27 DAYS) WITH NUMBER OF ROTATION INDICATED AT END OF CORRESPONDING VECTOR

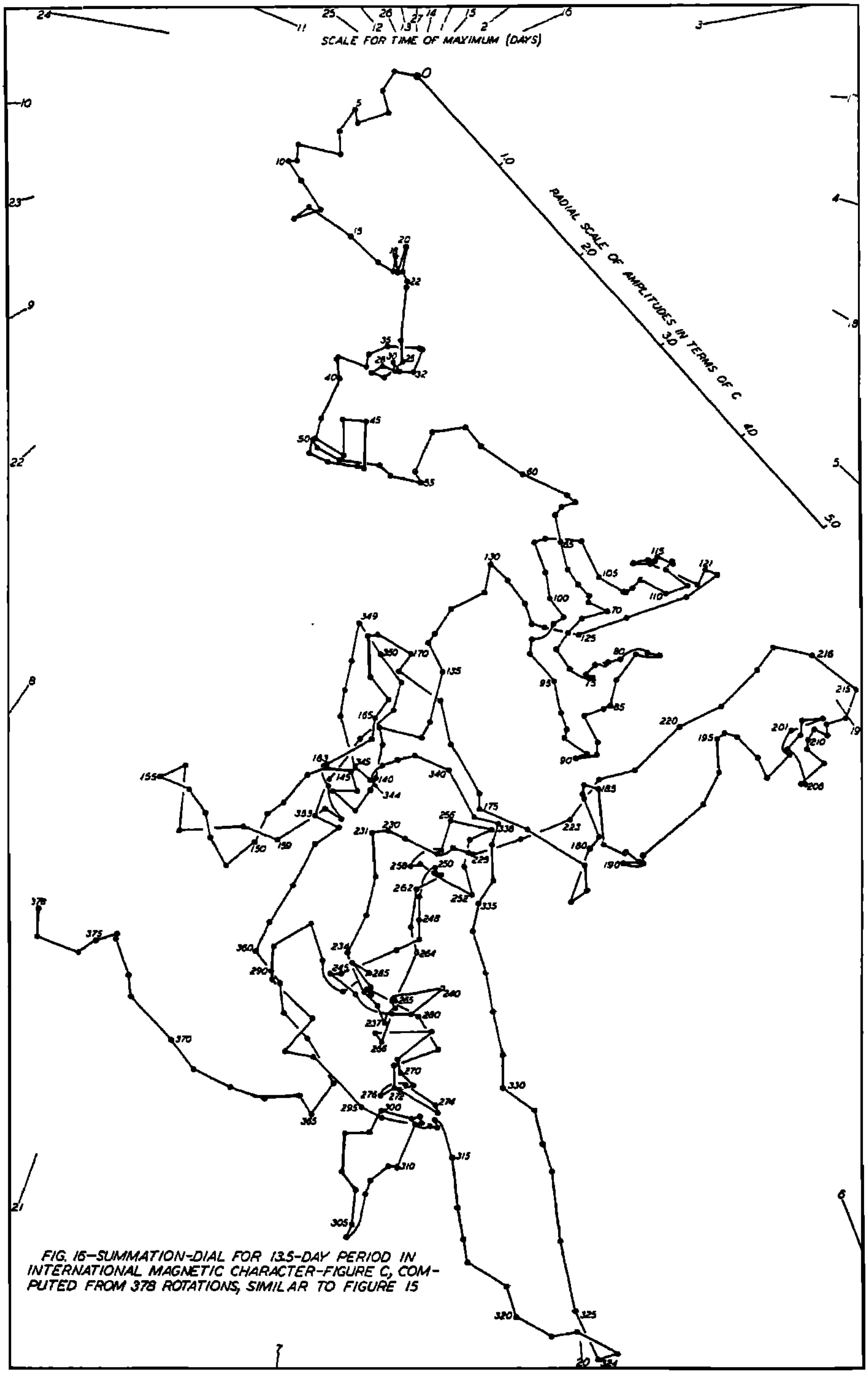


FIG. 16—SUMMATION-DIAL FOR 13.5-DAY PERIOD IN INTERNATIONAL MAGNETIC CHARACTER-FIGURE C, COMPUTED FROM 378 ROTATIONS, SIMILAR TO FIGURE 15

chance is so low in this case that even a multiplication by a few powers of ten (because of "selecting" this particular stretch from Figure 16, as mentioned in section 22) might not destroy the strong indication of a persistent wave of 13.5-day period within that interval of 25 rotations—vanishing, however, outside that interval.

Quasi-persistence is indicated in the summation-dials by sequences of vectors of approximately equal directions, for instance, the long sequences in the 13.5-day diagram between the points marked 216 to 231 (December 1921 to January 1923), or 324 to 338 (December 1929 to January 1931), which both correspond, of course, to distinct sequences in the former diagrams[34] for the 27-day recurrences. On the other hand, no such long sequences can be detected in certain parts of the summation-dials Figures 15 and 16, for instance, in the year 1926, rotations 272 to 284. These parts resemble closely the random walk pictured in Figure 8. And if we detect, in the random walk, Figure 8, the apparent "sequence" between the points 76 and 85, we are forced to give up the idea of distinguishing between random walk and quasi-persistence by a mere inspection of the summation-dial or haphazard considerations. In fact, the problem is to find a numerical measure for the geometrical property of the summation-dial which will give a clear distinction between random-walk conditions (Fig. 8), quasi-persistence (Figs. 15 and 16), and persistence (Fig. 11).

34. *Quasi-persistence measured by equivalent length σ of sequences*—In order to find such a measure, we consider a random walk, with the expectancy c of the single vectors. If we form the vectorial sum of every two successive vectors and divide it by 2, that is, if we form averages of every two successive vectors, we obtain a new set of vectors which has the expectancy $c/\sqrt{2}$: in general, if we average h successive vectors, these averages will have the expectancy $c/\sqrt{h}$ according to (19.1). If, however, we have a perfectly persistent wave without any superposed fluctuations, that is, if we have vectors of equal direction, the expectancy for the average of h successive vectors would be, of course, obtained as c.

We can express these conditions in another way. Suppose we compute, from N successive vectors given, the expectancies for the single vectors, for the averages of two vectors, etc., in general, for the averages of h successive vectors. [In order to be able to obtain a satisfactory approximation to the expectancy (section 27), the number of independent averages, roughly N/h, must not be too small; because of formula (27.1), h must not be much greater than about $N/50$, if we want the expectancy correct within 10 per cent.] We multiply the expectancy for the averages of h vectors by $\sqrt{h}$ [or, what amounts to the same, divide the expectancy for the sums of h vectors by $\sqrt{h}$], and obtain a value which we shall call $c(h)$. In the case of the random walk, $c(h)$ is always the same value $c=c(1)$, the expectancy for single vectors; but for persistent vectors, we obtain the ever-increasing value $c(h)=c\sqrt{h}$. In the case of quasi-persistence, $c(2)$ will be greater than $c(1)$, and $c(3)$ will be greater than $c(2)$, etc., but this increase will not continue proportionally to $\sqrt{h}$, as in the case of persistence, but will, in general, asymptotically approach an upper limit, limes $c(h)$, for $h=\infty$, say, $c(\infty)$.

If we now put $c(\infty)/c(1)=\sqrt{\sigma}$, we may call σ (which need not be an entire number), the *equivalent length of the sequences* of the quasi-persistent wave.

This designation of σ is justified as follows: In order to compute the expectancy for our quasi-persistent wave for large values of h, we can proceed as if the average of h single vectors, showing quasi-persistence, is the same as the average of (h/σ) random vectors of expectancy $c(1)$; in other words, as if, of the h vectors, every σ successive vectors are equal, and only h/σ vectors are independent. In fact, the average of (h/σ) random vectors has the expectancy $c(1)/\sqrt{h/\sigma}$, and multiplication with $\sqrt{h}$ gives $c(1)\sqrt{\sigma}$, that is, the same value $c(h)$ as actually computed.

Perhaps σ is the exact expression for what H. H. Turner[30] designated as a "chapter."

35. *Quasi-persistence in terrestrial-magnetic activity*—The actual computation for testing quasi-persistence was based on the summation-dials for the periods of 27, 13.5 and 9 days, the first two of which are reproduced in Figures 15 and 16. h was chosen equal to 4, 9, 16, and 25. Because of (17.7) the arithmetic mean of the amplitudes of a number of vectors, $(c'+c''+\ldots)/n$, distributed according to (17.3), is a constant fraction (0.8862) of the expectancy, defined as the square root of $[(c')^2+(c'')^2+\ldots]/n$. Now, as the law (17.3) can safely be taken as governing the distribution of the single vectors (Fig. 9) as well as their sums, we can be sure to make no systematic error in considering the ratios of the corresponding arithmetic means (which are somewhat easier to calculate) as sufficient approximations for the ratios of the expectancies $c(h)/c(1)$.

For instance, $c(4)/c(1)$ for the 13.5-day period was calculated as follows: The amplitudes of sums of four consecutive single vectors, namely, distances of the points marked *O* and 4, 4 and 8, . . . 372 and 376, as well as of the points 2 and 6, 6 and 10, . . . , 374 and 378, were measured on Figure 16. The sum of these 188 distances is 127.06 units of C, and the average distance therefore $127.06/188=0.6759C$. In order to deal with averages, and not sums, for four successive vectors, this value must be divided by 4, and then, in order to obtain the equivalent of $c(4)$ multiplied by $\sqrt{4}=2$. This gives $0.3380C$. This should, under random-walk conditions, be equal to the arithmetic mean of the lengths of the single vectors, that is, equal to the arithmetic mean of the $188\times 4=752$ distances 0 to 1, 1 to 2, 2 to 3, . . . , 375 to 376 and 2 to 3, 3 to 4, . . . , 377 to 378 (in this arithmetic mean, all single vectors appear twice except those for the rotations 1, 2, 377, and 378). On actual calculation,

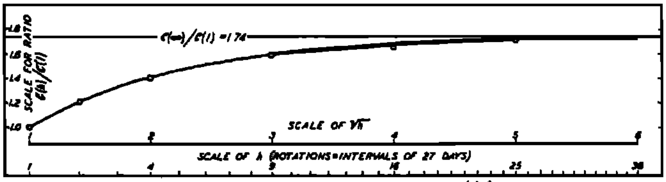


FIG. 17—QUASI-PERSISTENCE, INDICATED BY RATIO OF $c(h)$ (EXPECTANCY OF SUM OF h VECTORS DIVIDED BY $\sqrt{h}$) TO $c(1)$, FOR AVERAGE OF SINE-WAVES WITH PERIODS OF 27, 13.5, AND 9 DAYS IN THE INTERNATIONAL MAGNETIC CHARACTER-FIGURE C, 1906-1933; THE ASYMPTOTIC VALUE $c(\infty)/c(1)=1.74$ INDICATES EQUIVALENT LENGTH OF SEQUENCES $1.74^2=3.0$ ROTATIONS

this mean of these 752 distances is found to be only $0.2298C$. Consequently, our ratio $c(4)/c(1)$ is $0.3380/0.2298 = 1.470$. In the case of a perfectly persistent wave, we should have obtained for this ratio $\sqrt{4} = 2$, and in the other extreme, the random case, ratio 1.

Table 3 shows the results of the calculations. The average ratios have been entered against $\sqrt{h}$ as abscissae, in Figure 17, and fit well into an exponential curve which approaches asymptotically $c(\infty)/c(1) = 1.74$. Therefore, the equivalent length of sequences is $\sigma = 1.74^2 = 3.0$ rotations.

TABLE 3—*Quasi-persistence in the international magnetic character-figure C, 1906 to 1933*

Period	$c(1)$	Ratio $c(h)/c(1)$ for $h =$				
		1	4	9	16	25
days 27	$0.262C$	1.000	1.410	1.561	1.632	1.766
13.5	$0.264C$	1.000	1.470	1.729	1.713	1.738
9	$0.236C$	1.000	1.322	1.478	1.611	1.608
Average		1.000	1.401	1.589	1.652	1.704

In addition, for the periods of 27 and 13.5 days, the ratios have been computed for $h = 2$, for individual years. Only for the two years 1916 and 1917 (rotations 136 to 162), in the 27-day period, the ratio $c(2)/c(1)$ is smaller than 1, namely, 0.89 and 0.95; this is expressed in the summation-dial Figure 15, where the trace between the points marked 135 and 162 appears very irregular. For the 13.5-day period, the ratios for the same years are 1.20 and 1.12. Particularly high values of $c(2)/c(1)$, approaching the theoretical maximum of $\sqrt{2} = 1.41$ for persistence, are found, in the 27-day period, for the years 1911 (1.38), 1913 (1.39), and 1933 (1.38); for the 13.5-day period, for the years 1922 (1.37) and 1930 (1.40). The average ratio for all rotations (years 1906 to 1933) are $c(2)/c(1) = 1.196$ for the 27-day period, and 1.226 for the 13.5-day period; the average value 1.211 has been entered in Figure 17.

For $h = 4$, the ratios $c(4)/c(1)$ were computed for 14 pairs of years 1906 and 1907, etc. All ratios are greater than 1, the lowest being, of course, 1.04, in the 27-day period for the two years 1916 and 1917, which already gave the lowest ratios $c(2)/c(1)$. Particularly high values of $c(4)/c(1)$, approaching the value $\sqrt{4} = 2$ for persistency, are found, in the 27-day period, 1912 to 1913 (1.64), and in the years 1924 to 1925 (1.68), and, in the 13.5-day period, in the years 1930 to 1931 (1.80) and 1932 to 1933 (1.71).

In order to test the strong quasi-persistence noticeable in the former 27-day recurrence-diagram for the years 1928 to 1933,[34] the ratios $c(h)/c(1)$ were also computed for these years alone; they are, for $h = 2$, 4, and 9

For the 27-day period: $h = 2$, 1.252; $h = 4$, 1.436; $h = 9$, 1.661
For the 13.5 day period: $h = 2$, 1.308; $h = 4$, 1.698; $h = 9$, 2.151

The highest value, 2.151, would already correspond to an equivalent length of the sequences of more than $\sigma = 2.151^2 = 4.6$ rotations.

A diagram equivalent to Figure 17 may be called the *characteristic* for the period p in the observational material.

36. *Influence of quasi-persistence on tests for persistence: Effective expectancy*—The vectorial average of N vectors which have the expectancy $\mathbf{c}(1)$, and quasi-persistence characterized by the equivalent length σ of the sequences (N great compared with σ), has an expectancy which, multiplied by $\sqrt{N}$, we have called, in section 34, $\mathbf{c}(N)$; it is therefore $\mathbf{c}(N)/\sqrt{N}$, and if N is great enough so that $\mathbf{c}(N)/\mathbf{c}(1)$ has the limiting value $\sqrt{\sigma}$, we can write $\mathbf{c}(1)\sqrt{\sigma}$ for $\mathbf{c}(N)$ and obtain for the expectancy of the average of N vectors

$$\mathbf{c}(1)\sqrt{\sigma}/\sqrt{N} \tag{36.1}$$

This value differs from the random value $\mathbf{c}(1)/\sqrt{N}$, obtained by assuming successive vectors $\mathbf{c}$ independent, *by the factor* $\sqrt{\sigma}$. If, in quasi-persistent waves, we search for persistent waves as described in sections 28-32, *this* value (36.1) must be taken as the expectancy with which the amplitudes actually found by vector-addition must be compared. *The crucial ratio* κ *of the amplitudes actually found to their expectancy is therefore reduced in the ratio* $1/\sqrt{\sigma}$ *against the ratio calculated on the assumption of random-walk conditions, or of independence of successive single vectors, without regard to quasi-persistence.* The consequences for the considerations on the probability for chance $W(\kappa)$ are sometimes decisive, because even a small decrease in κ may mean a large increase of $W(\kappa)$, according to Table 2 (section 16). $\mathbf{c}(1)\sqrt{\sigma}$ may appropriately be called *effective expectancy*, as contrasted to the *ordinary expectancy* $\mathbf{c}(1)$.

The decisive influence (36.1) of quasi-persistence on tests for persistence as well as on the uncertainty of average sine-waves derived from a large material can also be expressed in another way: Against random conditions, the effective number (N/σ) of the available observations is reduced to $1/\sigma$ of its apparent number N.

We can now adjust our considerations in section 21. There, assuming random-walk conditions and starting from the ordinary expectancy $\mathbf{c}(1)$, we obtained, for the average vectors of the 27-day and 13.5-day periods for the 378 rotations 1906 to 1933, the values $\kappa = 2.49$ and 1.47, with $W(2.49) = 1/500$ and $W(1.47) = 1/9$. Taking $\sqrt{\sigma} = 1.74$ (which is certainly not too high, judged from Table 3), the consideration of quasi-persistence, as expressed in the effective expectancy, reduces κ to 1.43 and 0.84, raising $W(\kappa)$ to $W(1.43) = 1/8$ and $W(0.84) = 1/2$. These "probabilities for chance" are so high that there can be absolutely no doubt about the absence of a noticeable persistence in these periods of 27 and 13.5 days; or, expressed more accurately, if persistent waves of these periods existed, the material at hand is not sufficient to trace them.

In the same way, we can dispose of the persistent wave of 9.00-day period which Pollak[11] believed to have traced in the international magnetic character-figure C for the years 1906 to 1926. He obtained for this wave an average amplitude of $0.0412C$, and, with his expectancy, disregarding quasi-persistence, a value $W(2.76) = 1/2000$, which he considered sufficiently low to exclude pure chance. Our own calculation for the 9-day period gives, for the 284 rotations in the years 1906 to 1926, $c(1)=0.232C$ (only slightly different from the value $0.236C$ given, for all years 1906 to 1933, in Table 3); for $\sqrt{\sigma}$, we make, from Table 3, the conservative estimate 1.62 giving the effective expectancy $0.376C$. The expectancy for an average of 284 rotations is, therefore, $0.376C/\sqrt{284} = 0.0224C$, and $\kappa = 0.0412/0.0224 = 1.84$, with $W(1.84) = 1/30$, which is not at all suspiciously low. The full series 1906 to 1933 gives, by the way, about the same indication.

37. *Infection of adjacent periods by quasi-persistence*—We have seen that the 27-day sine-wave period in the international magnetic character-figure C shows quasi-persistence with $\sigma = 3.0$ rotations. It is easy to see that, for instance, the 28-day period must be affected by this quasi-persistence. Suppose, namely, the series of character-figures C to be divided into intervals of 28 days, beginning January 11, 1906. These single intervals would, on harmonic analysis, give amplitudes of 28-day sine-waves which are only little different from those of the 27-day sine-waves computed from the first 27 days in each 28-day interval; this is easily recognized by considering the folding process (section 10, and Fig. 3), in which the turning angles for the 27-day and 28-day periods, respectively, are $(360°/27) = 13°.33$ and $(360°/28) = 12°.86$, so that the successive links in the 28-day folding-process are only $0°.47$, $0°.94$, etc., less inclined against the vertical than the same links in the 27-day folding-process. Therefore, the ordinary expectancy $c(1)$ for the 28-day sine-waves will not differ greatly from that for the 27-day sine-waves. However, successive 28-day intervals begin always one day later than the successive 27-day rotation; if, in the summation-dial for sine-waves of 27-day period, the vectors for successive rotations have nearly the same phase, because of quasi-persistence, then, in the summation-dial for the 28-day sine-waves, because of the relative shift of 27-day and 28-day intervals, the vectors should have phases increasing about $(360°/27) = 13°$ from one 28-day interval to the next [the maxima appearing to occur one day earlier in successive 28-day intervals]. Therefore, the general aspect of the summation-dial for 28-day sine-waves would be about the same as that for the 27-day sine-waves (Figure 15), except that successive vectors were turned by about 13° anti-clockwise. This would not greatly affect the values of $c(2)/c(1)$ and even $c(3)/c(1)$ (sections 34, 35); only for higher values of h, $c(h)/c(1)$ for the 28-day period may not increase to the same values as given in Table 3, so that the equivalent length σ of sequences for the 28-day period may be smaller than 3.0. In other words, the 28-day period will show quasi-persistence because it is "*infected*" by the quasi-persistence of the 27-day period. This makes it, apart from other reasons,[53] difficult to determine the exact length of the 27-day recurrence-interval, which might differ from 27 days by a few tenths of a day, and could be recognized as yielding the highest value of σ.

This kind of infection will diminish the greater the difference of the periods. For periods of 30 days, for instance, the infection by the 27-day period will be small. Actual calculation of 27-day periods and 30-day periods for the character-figure C for the two years 1924 and 1925 gave the following results—the values for 27-day period and 30-day period

[53]J. M. Stagg, Meteorological Office, Geophysical Mem. No. 40, London (1927).

being, in each case, printed after each other: Number of full intervals considered 27, 24; expectancy for single interval $c(1)=0.258C$, $0.248C$; $c(2)/c(1)=1.26$, 1.22; $c(4)/c(1)=1.68$, 1.21. This sample calculation allows one to assume, for the 30-day period for all years 1906 to 1926 used by Pollak, $\sqrt{\sigma}$ about 1.2, and the ordinary expectancy $c(1)=0.252C$, namely, $0.010C$ less than $c(1)=0.262C$ for the 27-day period in the same years 1906 to 1926, so that the effective expectancy becomes $c(1)\sqrt{\sigma}=0.302C$.

The expectancy for the average wave of Pollak's 255 intervals of 30 days is therefore $0.302C/\sqrt{255}=0.0189C$, and the same value will hold, with high approximation, for a wave of 29.9 days. The actually calculated sine-wave of Pollak for 29.9 days from his material has an amplitude of $0.0511C$. Therefore, $\kappa=(0.0511/0.0189)=2.70$, with $W(2.70)=1/1460$. This value might look suspiciously small, though not so small as the value of 1/110,000 which Pollak himself derives using a too low expectancy. But, since the 29.9-day period is picked, *because* of its high amplitude, out of 73 amplitudes actually calculated, the "probability for chance," according to section 22, is $(73/1460)=1/20$, and this is not small enough to warrant the definite assumption of persistence. There remains the possibility of long-range quasi-persistence, corresponding to Ad. Schmidt's idea of deep-seated long-lived foci in the Sun's surface-layers, with a rotation of about 30-day period.

38. *Interference*—The infectiousness of quasi-persistence, as described in section 37, is related to the general interference-phenomenon leading to the "*spurious periodicities*" which A. Schuster[8] discovered as complete analogues to the secondary maxima obtained in analyzing homogeneous light by a spectroscope of finite resolving power. His exact formula will be illustrated here by a straightforward application of the summation-dial which will yield an approximation sufficient for practical use.

Consider the persistent sine-wave of half-year period (section 29) in the international magnetic character-figure C, of amplitude $0.0675C$, and constant phase. The summation-dial of this period for the 56 half-years 1906 to 1933, freed from the irregular fluctuations exhibited in

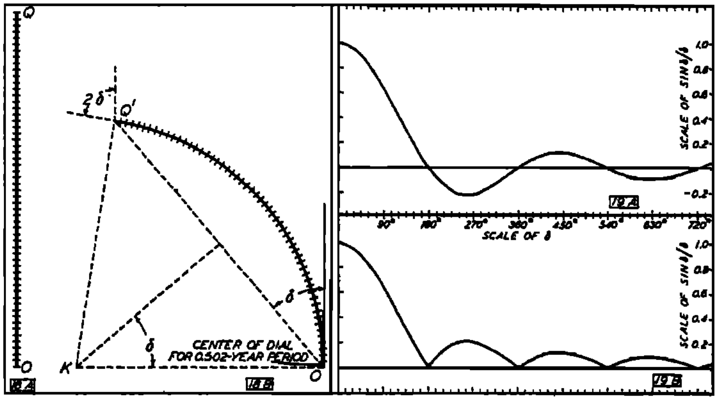


FIG. 18—INTERFERENCE, DEMONSTRATED IN SUMMATION-DIALS FOR 56 PERIODS OF 0.500 YEAR (18 A), AND 56 PERIODS OF 0.502 YEAR (18 B)
FIG. 19—FUNCTION SIN δ/δ

Figure 11, would be a succession of 56 perfectly aligned single vectors leading from the center O of the dial to the point Q (Fig. 18*A*).

What should we obtain if we would analyze the train of these 56 sine-waves of exactly $p=0.500$-year period for a slightly different period of length, say, $p'=0.502$ year? Fifty-six complete waves of period p' would cover an interval of 56.112 half-years. The harmonic dial for the 56 single waves of period p' in that interval would show amplitudes practically equal to that of the actual half-year period, but the maxima of each wave of period p' would shift gradually and occur earlier. Since $250\ p'=251\ p$, the phases of two periods would agree again after 250 intervals of length p'; consequently, the phase of the period p' shifts from one interval of length p' to the next by $(360°/250)=1°.44$. After 56 intervals, the phase-shift would be about 81°. We will call this angle 2δ. The summation-dial for the period $p'=0.502$ year is therefore approximately part of a circle (Fig. 18*B*), the length of the arc $O'Q'$ being equal to OQ, and the tangent of the circle drawn in Q' forming an angle of $2\delta=81°$ with the tangent drawn in O'.

Now the vectorial sum of the 56 sine-waves with period p' is represented by the straight line $O'Q'$, while the vectorial sum of the 56 sine-waves with period p is represented by OQ, which, as we have seen, is equal to the length of the arc $O'Q'$. The amplitudes of the average vectors of periods p and p' are obtained by dividing the vectorial sums by 56. Therefore, the ratio of the amplitudes of the average sine-waves with periods p' and p is equal to the ratio of the lengths of the chord and the arc $O'Q'$, or $\sin\delta/\delta$, as the auxiliary construction in Figure 18*B* indicates.

In general, consider a train of waves with a persistent period of length p, and suppose it to be analyzed for a period of slightly different length, $p'=p+\Delta p$, where $\Delta p/p$ is small. Putting $m=p/\Delta p$, we find $mp'=(m+1)p$, and since m is large, this will hold also if, instead of the exact value $m=p/\Delta p$, we take the nearest integer for m. Then our equation means that the interval covered by m periods p' is covered by $(m+1)$ periods p. This means that, in the summation-dial for period p', the shift of phase between a certain vector to the vector for an interval occurring mp' later is 2π, and therefore the phase-shift for successive vectors is $2\pi/m$. If the interval considered contains only n waves of period p, the angle 2δ of Figure 18*B* becomes $2\delta=2\pi n/m=2\pi n\ \Delta p/p$, and the ratio of the average amplitude of the "spurious" period p' to the average amplitude of the persistent wave with period p becomes

$$\sin\delta/\delta \text{ with } \delta=\pi n\ \Delta p/p \tag{38.1}$$

where p is the length of the exact period, $(p+\Delta p)$ the length of the spurious period, and np the length of the whole interval analyzed. Because of the slight idealization assumed at the end of the summation-dial, this formula gives the correct value within the limit $1/n$, which is practically sufficient since n must be large enough anyway (see footnote 46).

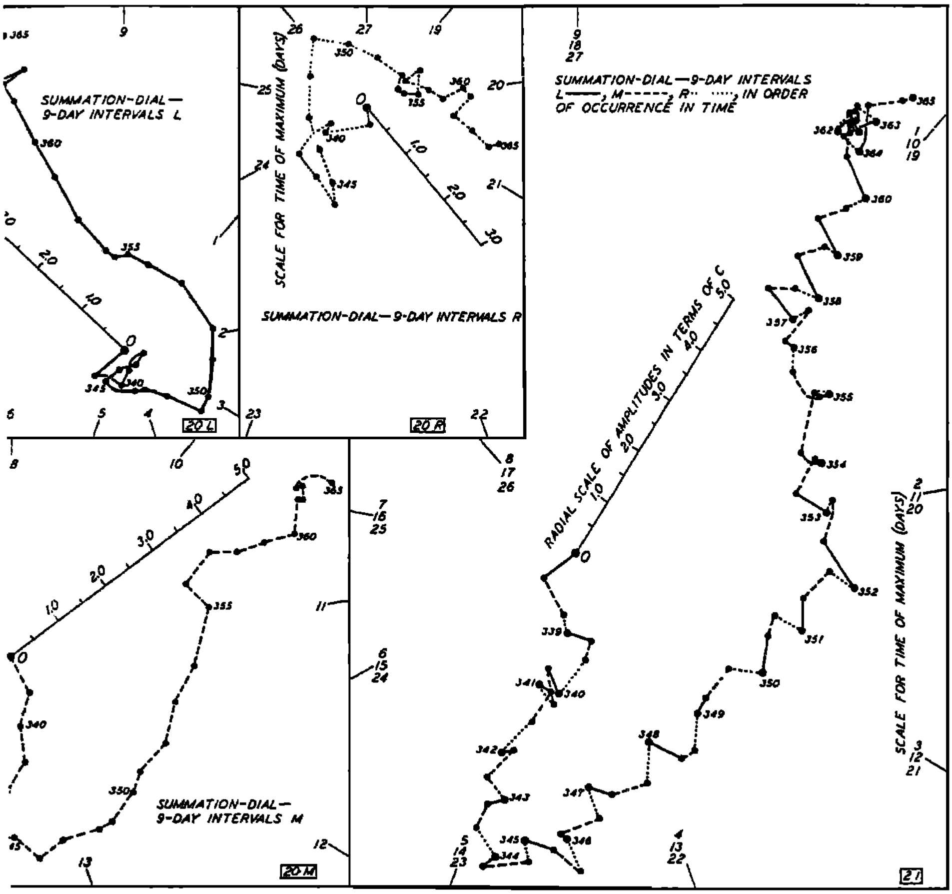
The function $\sin\delta/\delta$ has been plotted in Figure 19*A*. The "spurious" amplitude vanishes for $\delta=\pi$, 2π, etc., that is, $\Delta p=p/n$, $2p/n$, etc. This is only another expression for the independence of the harmonic coefficients in the series (5.2), because the length of the whole interval is np, so that the period p has the frequency n (supposed to be high), and the frequencies $(n-1)$, $(n-2)$, ... have the periods $[np/(n-1)]$, $[np/(n-2)]$, ..., or, nearly, $[p+(p/n)]$, $[p+(2p/n)]$, etc. Of course, our discussion applies equally to periods $(p-\triangle p)$, etc., so that Figure 19 may be extended symmetrically to $\delta=0$. The negative sign of $\sin \delta/\delta$ between $\delta=\pi$ and 2π, 3π and 4π, etc., is, in our case, not significant and can be disregarded, as in Figure 19*B*.

If the observational material contains a persistent wave, the period p of which is no entire submultiple of the total interval T of observation, ordinary harmonic analysis would, in the series (5.2), not indicate the full amplitude c of this wave, but only a part of it in spurious periodicities. For instance, if $(n+0.5)p=T$, where n is an integer, we have $\triangle p/p=0.5/n$, and $\delta=\pi/2$, so that the waves with frequencies n and $(n+1)$ would show amplitudes with $0.637c$. It is therefore necessary to search the neighborhood of periods with suspiciously high amplitudes for the exact period of a possibly persistent period. This is done by Darwin's method of approximation, first used for the calculation of tides, and described by Stumpff and Pollak (foot-notes 10 and 11). The summation-dial, Figures 18*A* and 18*B*, is a reliable guide in applying this method, which approximates the arc $O'Q'$ in Figure 18*B* by a number of smaller chords: in other words, partial average vectors for the period p are calculated for a small number of groups—for instance, 16 groups of 4 half-years in our case—and these partial averages are combined in different ways, namely, *without* phase-shift to obtain the average vector for period p, and *with* appropriate phase-shifts to obtain the average vector for adjacent periods $(p+\triangle p)$.

For another aspect of interference, see appendix section *A8*.

39. *Special kinds of quasi-persistence*—The interference-phenomenon described in section 38 can be conceived as some regular kind of quasi-persistence caused, by a persistent wave of period p, in waves with adjacent periods $(p+\triangle p)$. The diagram of Figure 17 for such an adjacent period would begin with interference "beats" similar to those of Figure 19*B*, but these oscillations would gradually be damped and end in the value $c(h)/c(1)$ for the actual quasi-persistence. A lunar wave derived from material with larger solar waves, for instance, in atmospheric pressure, or in the terrestrial-magnetic force, is a typical example.

A more general kind of quasi-persistence would be given by a case where, in the summation-dial, not the successive intervals, but perhaps the first, fourth, seventh, etc., exhibit a tendency to have the same phase. We obtain exactly this case, if we break up our 378 rotations of 27 days in the international magnetic character-figure C into $(3\times378)=1134$ intervals of 9 days. We shall designate each of these 9-day intervals by the rotation-number and distinguish the three thirds of each rotation by the letters L, M, R (left, middle, right). Figures 20 and 21 show, in different arrangement, the summation-dials for these sine-waves with 9-day periods for the years 1931 and 1932, comprising the 27 rotations Nos. 339 to 365, or 81 intervals of 9 days. Figure 21 shows the 81 vectors for the 9-day intervals as they follow each other in time, that is, for interval 339L, 339M, 339R, etc., up to 365R. The vectors for intervals L, M, R have been distinguished by drawing them in full, dashed, and dotted, respectively. In Figure 21 the number of each rotation is entered against the end (marked more boldly) of the last



0—SUMMATION-DIALS FOR 9-DAY PERIOD SINE-WAVES IN INTERNATIONAL MAGNETIC CHARACTER-FIGURE C, AND 1932, AS COMPUTED FOR 9-DAY INTERVALS L (DAYS 1–9), M (DAYS 10–18), R (DAYS 19–27), OF 27-DAY INTERVALS, ROTATIONS 339–365

1—SUMMATION-DIAL L-M-R FOR 9-DAY INTERVALS OF FIGURE 20

vector R for each rotation. (It is to be remarked that should we retain only these points marked in Figure 21 and omit the points for the ends of vectors L and M, we should obtain the summation-dial, on a three-fold magnified scale, for the 9-day periods calculated from *whole* rotations of 27 days, analogous to Figures 15 and 16, and discussed in section 35.) The time of maximum is indicated by the scales around the borders of the dials, by days 1 to 9 for L, 10 to 18 for M, 19 to 27 for R, according to the numbering of the days in the rotations.

Calculations similar to those in section 35, based on Figure 21, lead to the expectancy, for single vectors, $c(1) = 0.540C$, and for the ratios $c(h)/c(1)$, we obtain 1.109 for $h=3$, 1.369 for $h=6$, and 1.669 for $h=9$.

On inspection of Figure 21, we find the feature indicated above, namely, practical independence for vectors immediately following each other, but every third vector—for instance, those for M-intervals indicated by dashed lines—has a tendency to keep its phase. This is brought out more clearly in Figures 20L, 20M, and 20R, where the vectors for the L-, M-, and R-intervals are added separately and show considerable quasi-persistence. If we calculate $c(h)/c(1)$ for these three summation-dials of Figure 20, we obtain, on the average, 1.241 for $h=2$ and 1.395 for $h=3$. The contrast between the values of $c(h)/c(1)$ for $h=3$, namely, 1.109 for Figure 21, 1.395 for Figures 20L, 20M, and 20R, is the numerical expression for this particular kind of quasi-persistence, which could be called *intermittent*: The three thirds L, M, R of each rotation give nearly independent 9-day sine-waves, but corresponding thirds, for instance, the L-intervals alone, show strong quasi-persistence. This proves incidentally that the 9-day period has no self-existence, but is only a sub-period of the 27-day period, which is the actual periodicity.

The value $c(3)/c(1)=1.109$ for Figure 21, small as it is, is nevertheless greater than unity and reveals a weak degree of quasi-persistence, which, however, seems to be a general phenomenon in many cases in which dependent ordinates are divided into sets, because, for instance, a group of high ordinates, divided up by a limit between two sets, adds likewise, in both sets, to the cosine-coefficient a_ν of the successive sets.

The most general definition of quasi-persistence as distinguished from random conditions leads, of course, to the same fundamental difficulties encountered in a satisfactory definition of the term "accidental" in the theory of probability. In this respect, we refer to the books of Mises and Kamke[6], especially to the definition of the "fields of probabilities" discussed by Kamke.

From our much-used example of the 378 rotations in C, we can easily construct an illustration of these remarks. Imagine our 378 rotations divided into seven groups comprising rotations Nos. 1 to 54, 55 to 108, 109 to 162, . . . , and 325 to 378. In each group, mix the numbers of the rotations at random. Then draw the summation-dial for the 27-day period which, in the points 54, 108, . . . , 378 would be identical with that in Figure 15, and test for quasi-persistence. Obviously we should obtain a different curve from Figure 17, namely, $c(h)/c(1)$ would remain near unity for low values of h, because the mixing has produced random conditions for these, but with h approaching 54, $c(h)/c(1)$ will rise to the limiting value indicated in Figure 17.

If a persistent wave of amplitude c is present, $c(h)$ in Figure 17, with abscissa $\sqrt{h}$, would finally approach the line $c\sqrt{h}$, or $c\sqrt{N}$, if we write N for large values of h. The κ-test for persistent waves (section 36) can easily be applied to this characteristic, because $c\sqrt{N}/c(1)\sqrt{\sigma}=c/(c(1)\sqrt{\sigma}/\sqrt{N})$, and this is κ because of (36.1). We can therefore enter a uniform scale of κ, where $\kappa=1$ corresponds to the effective expectancy $c(1)\sqrt{\sigma}$ (see section 41).

40. *Periodicities of other form than sine-waves*—The application of harmonic analysis to research on geophysical periodicities is sometimes criticized because the form of the periodicity—for instance, an average diurnal variation of magnetic declination—is said to be in no way connected with sine-waves, which are forced upon it by the purely mathematical process of harmonic analysis. In fact, a *physical* reason for expecting periodicities having the form of sine-waves is given only in a

few cases, for example, if the phenomenon is due to resonance in an oscillating system (semidiurnal wave of atmospheric pressure), or if it is caused by forces changing like sine-waves (tides), or wherever a differential equation of the type $y'' = -ky$ may hold. But a *mathematical* reason for applying harmonic analysis is always given, because it furnishes an adequate approximation, replacing the given ordinates by a few harmonic coefficients.

In order to meet the criticism mentioned, the 27-day recurrence-phenomenon in magnetic activity has been treated here as an example *just because* the harmonic analysis is, in this case, a mere mathematical affair, with no simple physical meaning ascribable to each of the separate sine-waves of 27-, 13.5-, and 9-day periods. Yet we have been able to develop the ideas of persistence and quasi-persistence in this material. There can be therefore no doubt that the same methods can be successfully applied in dealing with other geophysical and meteorological phenomena.

However, the following outline of a test for persistence, quasi-persistence, or random fluctuations will show how our methods can be generalized so that they do no more imply an explicit reference to harmonic analysis. Consider the international magnetic character-figure C, 1906 to 1933, arranged in 378 rotations, that is, written in 27 columns with 378 rows. Form, for each rotation, the standard deviation, take its square, sum up for all rows, divide by 378, take square root: the value obtained is called $\zeta(1)$. Add each two successive rows of C and divide by two, thus obtaining average 27-day variations for two rotations each. For these new average rows, form standard deviation, take its square, sum up, divide by number of average rows, take square root, multiply with $\sqrt{2}$; the value obtained is called $\zeta(2)$. In general, form average rows of h successive rotations, compute standard deviation for each row, square, sum up, divide by number of average rows, take square root, multiply by $\sqrt{h}$, so obtaining values called $\zeta(h)$. With random fluctuations, $\zeta(h) = \zeta(1)$; with persistent periodicities $\zeta(h) = \zeta(1)\sqrt{h}$; with quasi-persistent periodicities of 27 days, $\zeta(h) \rightarrow \zeta(1)\sqrt{\sigma}$, where σ is the equivalent length of sequences.[54]

Remembering formula (11.6), and the conception of the generalized harmonic dial (section 13), in which the vector is proportional to ζ, it is easily verified that this method corresponds exactly to the generalization of the two-dimensional summation-dial to the generalized harmonic dial. Instead of Figures 15 and 16, we should consider a track of vectors, a summation-dial, in 26 dimensions. In the case of the character-figure C, the value for σ obtained will be not far from that of $\sigma = 3.0$ rotations obtained from the three sine-waves of 27-, 13.5-, and 9-day periods, since the amplitudes of these waves contribute most of $\zeta(1)$.

In general, rows of r ordinates would be written down, and averages of h such rows formed. In particular, for $r = 1$, we should obtain some form of Lexis test of independence of successive ordinates, suitable for geophysical applications.

41. *General statistical test for periodicity in geophysical phenomena*—

[54]As to persistent waves, the method given above is materially the same as that given in 1924 by E. T. Whittaker and G. Robinson (see foot-note 35). It had been independently found and applied in the author's doctor thesis, Göttingen, 1922.

In short, the full test of a period p for quasi-persistence and persistence is obtained as follows: Divide, in a suitable way as shown in section 35, the whole interval T of observations into intervals of equal length hp. Compute the amplitudes of the sine-wave of period p for each interval and from these amplitudes compute their expectancy according to (24.1). Multiply this expectancy by $\sqrt{h}$ and obtain $c(h)$. Derive $c(h)$ for various values of h, beginning with $h=1$, and ending with a value of h so that hp is still only about 1/20 of T, so that the function $c(h)$, represented as ordinate against the abscissa $\sqrt{h}$, is properly determined. From this characteristic, the nature of the fluctuations can be judged. Instead of the amplitudes of sine-waves, standard deviations can be used as indicated in section 40.

In Figure 22, five typical cases are shown. They will be enumerated below, and we add tentatively a few more examples for each type of the characteristic:

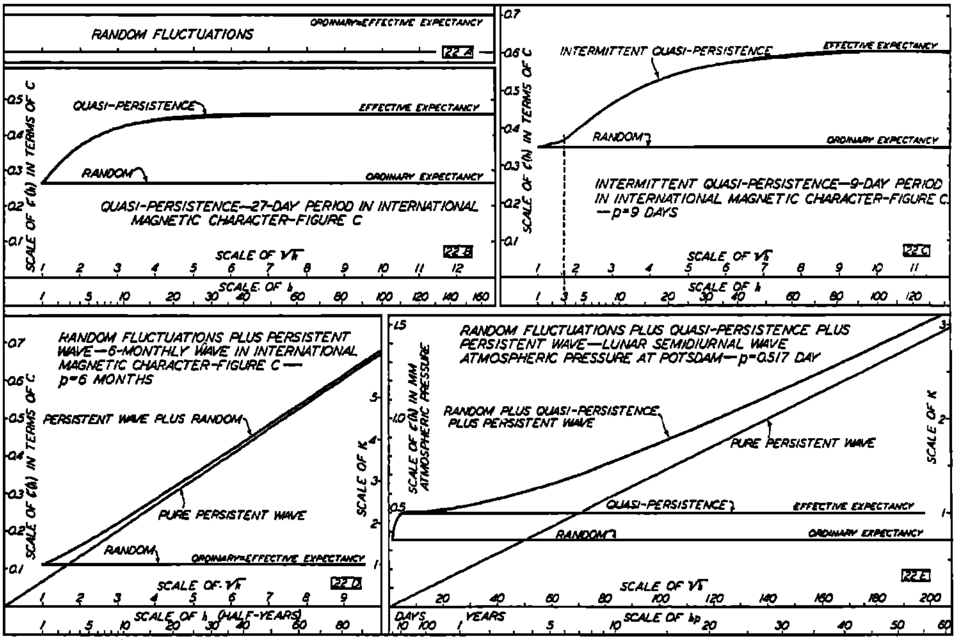


FIG. 22—$c(h)=\sqrt{h}$ TIMES ORDINARY EXPECTANCY FOR AVERAGE SINE-WAVES OF PERIOD p RESULTING FROM HARMONIC ANALYSIS OF INTERVALS OF LENGTH hp AS FUNCTIONS OF $\sqrt{h}$ —FIVE TYPICAL CASES

(*A*) *Random fluctuations—$c(h)$ equal to the ordinary expectancy $c(1)$—* Disintegrations of radioactive substances; artificial examples obtained by random sampling (summation-dials resembling Figs. 7 and 8).

(*B*) *Random fluctuations plus quasi-persistence*—$c(1)$ increases asymptotically from the ordinary expectancy $c(1)$ to a limiting value, the effective expectancy $c(1)\sqrt{\sigma}$ (summation-dials resembling Figs. 15 and 16). 27-day period in terrestrial-magnetic activity (section 35), aurora, and in solar phenomena, due to solar rotation; many meteorological phenomena, for instance, the periods of from 20 to 40 days in atmospheric pressure (Weickmann), the waves of periods of a few days in rainfall (Defant), the period of 3.5 years in atmospheric pressure in the Indian Ocean, the cycles of Brückner, A. E. Douglass (tree-rings), and many others.

(*C*) *Intermittent quasi-persistence*—$c(h)$ increases slowly for $c(1)$ up to $c(h_0)$, from there follows example *B* (summation-dials resembling Fig. 21). Period p is submultiple of actual period of length h_0p. Nine-day sine-waves in international magnetic character-figure *C* (section 39), and all cases of subperiods.

(*D*) *Random fluctuations plus persistence*—$\underline{c(h)}$ increases from $c(1)$, approaching asymptotically the straight line $c\sqrt{h}$, where c is the amplitude of the persistent wave. Probability of chance for persistent wave judged by $W(\kappa)$, (17.6), with κ = ratio of $c(h)$ to ordinary expectancy $c(1)$ (summation-dials resembling Fig. 11). Six-monthly wave in terrestrial-magnetic activity (section 29); the period of about 11 years in sunspots (?) and its effects in geophysical phenomena; most annual variations in meteorology; cyclic variations in the radiation of variable stars.

(*E*) *Random fluctuations plus quasi-persistence and persistence*—Combination of *B* or *C* with *D*. Probability of chance for persistent wave judged by $W(\kappa)$, (17.6), with κ = ratio of $\underline{c(h)}$ to effective expectancy $c(1)\sqrt{\sigma}$. All waves of 24 solar- and lunar-hour period, and their subperiods, in terrestrial magnetism, atmospheric electricity, meteorology, etc. Periods in sunspots other than 11 years. Biological and economical cycles. Quasi-persistence exhibited in the vectorial differences between the waves for single intervals and the persistent wave.

The illustration of these five cases in the *summation-dial* may finally be indicated: *A*—Random-walk; *B*—Modified random walk, so that each successive direction has a preference for the direction of the last vector; *C*—Like *B*, but the preferred direction is, for instance, that of the third vector before; *D*—Modified random walk preferring a *fixed* direction; *E*—Combination of *B*, or *C*, with *D*.

42. *Acknowledgments*—The numerical and graphical examples given in this paper were worked out with assistance given by C. C. Ennis and W. C. Hendrix at Washington, D. C., and by W. Zick at Eberswalde, Germany.

APPENDIX

A1. Harmonic analysis of equidistant ordinates: *Theorem I*—The interval $x=0$ to 2π is divided into r equal intervals, of length $2\pi/r$, by the abscissae $0, x_1, x_2, \ldots x_r$, where $x_\rho=\rho\, 2\pi/r$. A function $f(x)$ is given by the ordinates $y_\rho=f(x_\rho)$ for $\rho=1, 2, \ldots r$. The arithmetic mean of the y_ρ may be $f_0=\Sigma y_\rho/r$; their standard deviation may be ζ, defined by $\zeta^2=\Sigma(y_\rho-f_0)^2/r$. Consider a sum of sine- and cosine-functions of frequency $\nu=0, 1, \ldots, k$, with $k<r/2$

$$(A1.1)\qquad \phi_k(x)=a_0+\sum_{\nu=1}^{k}(a_\nu\cos\nu x+b_\nu\sin\nu x)$$

The coefficients a_0, a_ν, b_ν ($\nu=1, 2, \ldots k$) of $\phi_k(x)$ must be determined so that $\phi_k(x)$ approximates the given ordinates y_ρ of $f(x)$ in the meaning of least-square adjustment, that is, so that the mean square s_k^2 of the residuals $y_\rho-\phi_k(x_\rho)$, that is

$$(A1.2) \qquad s_k^2=\Sigma_\rho\,[y_\rho-\phi_k(x_\rho)]^2/r$$

has the minimum value possible. The solution is

$$(A1.3) \qquad a_0=\sum\nolimits_{\rho=1}^{r} y_\rho/r=f_0,$$

$$a_\nu=\sum\nolimits_{\rho=1}^{r} y_\rho \cos \nu x_\rho/(r/2),$$

$$b_\nu=\sum\nolimits_{\rho=1}^{r} y_\rho \sin \nu x_\rho/(r/2)$$

a_ν, b_ν are called harmonic coefficients. If r is an even number, a final term $a_{(r/2)}\cos (r/2)x$ can be added to $\phi_k(x)$, for which the minimum condition $(A1.2)$ gives

$$(A1.3a) \qquad a_{(r/2)}=(-y_1+y_2-y_3+y_4-\ldots+y_r)/r$$

Furthermore, the ordinates $\phi_k(x_\rho)$ of the approximating function have the average value

$$(A1.4) \qquad a_0=\sum\nolimits_{\rho=1}^{r} \phi_k(x_\rho)/r$$

and their standard deviation η_k is given by

$$(A1.5) \qquad \eta_k^2=\sum\nolimits_{\nu=1}^{k} (a_\nu^2+b_\nu^2)/2$$

or, for r even and $k=r/2$

$$(A1.5a) \qquad \eta_{(r/2)}^2=\sum\nolimits_{\nu=1}^{(r/2)-1} (a_\nu^2+b_\nu^2)/2+a_{(r/2)}^2$$

Finally for $k<r/2$

$$(A1.6) \qquad s_k^2=\zeta^2-\eta_k^2=\zeta^2-\sum\nolimits_{\nu=1}^{k} (a_\nu^2+b_\nu^2)/2$$

If the number of coefficients a_0, a_ν, b_ν in ϕ_k equals the number r of ordinates [for r even, $k=r/2$, for r uneven, $k=(r-1)/2$], the ordinates y_ρ are represented exactly by the ordinates of ϕ_k.

Proof—The proof is based on the fact that the system of functions 1, $\cos \nu x$, $\sin \nu x$ ($\nu=1$ to k) are orthogonal[55] in the interval $x=0$ to 2π. This fundamental property is expressed in the following formulae, in which the sums are extended from $\rho=1$ to r and the indices ν and μ range between 1 and $(r-1)/2$, for r uneven, and between 1 and $r/2$ for r even, unless the index $r/2$, for r even, is expressly indicated by a separate formula.

$$(A1.7) \qquad \Sigma \cos \nu x_\rho=0\ (\nu<r),\ \Sigma \sin \nu x_\rho=0\ (\nu \leqq r)$$

$$(A1.8) \qquad \Sigma \cos^2 \nu x_\rho=r/2,\ \Sigma \sin^2 \nu x_\rho=r/2\ (\nu<r/2),\ \Sigma \cos^2 (r/2)\,x_\rho=r$$

[55]Developments of arbitrary functions into series of orthogonal functions, such as sine-waves, spherical harmonics, etc., are discussed, for instance, in R. Courant and D. Hilbert, Methoden der mathematischen Physik, 2nd ed, Berlin, 1931.

$$(A1.9) \qquad \Sigma \cos \nu x_\rho \cos \mu x_\rho = 0,\ \Sigma \sin \nu x_\rho \sin \mu x_\rho = 0 \quad (\nu \neq \mu)$$

$$\Sigma \cos \nu x_\rho \sin \mu x_\rho = 0 \quad (\nu = \mu \text{ or } \nu \neq \mu)$$

We first prove ($A1.7$) by Moivre's theorem, namely, (writing exp z for e^z) $\cos \nu x_\rho + i \sin \nu x_\rho = \exp i\, \nu x_\rho = \exp i\, \nu \rho 2\pi/r = (\exp i\, \nu\, 2\pi/r)^\rho = q^\rho$, putting $q = \exp i\, \nu\, 2\pi/r$. Summing these equations from $\rho = 1$ to r, we obtain on the left hand $\Sigma \cos \nu x_\rho + i \sin \nu x_\rho$, and on the right hand the geometrical series

$$(A1.10) \qquad q + q^2 + q^3 + \ldots + q^r = q(q^r - 1)/(q-1)$$

But this is zero, for the denominator $(q-1) \neq 0$, since $1 \leqq \nu < r$ and $q^r = \exp i\, \nu 2\pi = 1$. The real and the imaginary part of the left-hand side must therefore also vanish, proving ($A1.7$).

The well-known formulae for $\cos (\nu + \mu)x$, $\cos (\nu - \mu)x$, etc., give at once for all values of ν and μ

$$(A1.11) \qquad \begin{cases} 2 \cos \nu x_\rho \cos \mu x_\rho = \cos (\nu + \mu) x_\rho + \cos (\nu - \mu) x_\rho \\ 2 \sin \nu x_\rho \sin \mu x_\rho = \cos (\nu - \mu) x_\rho - \cos (\nu + \mu) x_\rho \\ 2 \cos \nu x_\rho \sin \mu x_\rho = \sin (\nu + \mu) x_\rho - \sin (\nu - \mu) x_\rho \end{cases}$$

Summing from $\rho = 1$ to r, the right-hand sums vanish because of ($A1.7$), except $\Sigma \cos (\nu - \mu) x_\rho = r$ for $\nu = \mu$, because $\cos 0 = 1$, and $\Sigma \cos (\nu + \mu) x_\rho = r$ for $\nu = \mu = r/2$, r even, because $\cos r\, x_\rho = \cos 2\pi = 1$. That proves ($A1.8$) and ($A1.9$)[14].

We can now prove our theorem. First, ($A1.4$) follows from ($A1.7$), and if we form $(\phi_k(x_\rho) - a_0)^2 = (a_1 \cos x_\rho + b_1 \sin x_\rho + \ldots + a_k \cos kx_\rho + b_k \sin kx_\rho)^2$ and sum from $\rho = 1$ to r, ($A1.5$) follows from ($A1.8$) and ($A1.9$). We consider now a function $\phi_k^*(x)$ of the same form as $\phi_k(x)$, but with arbitrary coefficients a_0^*, a_ν^*, b_ν^*, while for $\phi_k(x)$ we take the coefficients defined by ($A1.3$). We consider the sum $\phi_k(x) + \phi_k^*(x)$ and shall find that this approximation to y_ρ is worse than that given by $\phi_k(x)$ alone, unless all coefficients of $\phi_k^*(x)$ disappear. We form the square of the residual (omitting the index k in $\phi_k(x_\rho)$ and $\phi_k^*(x_\rho)$ where it is not necessary)

$$[y_\rho - (\phi(x_\rho) + \phi^*(x_\rho))]^2 = y_\rho^2 + \phi(x_\rho)^2 + \phi^*(x_\rho)^2 - 2\, y_\rho \phi(x_\rho) - 2\, y_\rho \phi^*(x_\rho) + 2\, \phi(x_\rho)\phi^*(x_\rho)$$

Inserting the series ($A1.1$) for ϕ and its equivalent for ϕ^*, and adding

[14]All these formulae can easily be transformed into simple geometrical problems by means of the harmonic dial or our folding process (section 10). ($A1.7$) is, for instance, only the expression for the closing of a regular polygon, if star-shaped polygons are admitted. ($A1.8$) and ($A1.9$) refer to some kind of epicyclic motion, described by a point on the circumference of a circle revolving with frequency $(\nu + \mu)$, while its center revolves with frequency $(\nu - \mu)$ on another circle of the same radius and fixed center. This explains the regularity of Figure 4. With k circles with radii c_ν, each center moving with frequency ν on the circumference of the preceding circle, with beginning of the movement given by the phases α_ν, the movement of a point on the circumference of the outermost circle, projected on a vertical line, reproduces the function $\phi_k(x)$; this is the principle of tidal computing machines.

Of course, the harmonic dial is equivalent to the ordinary geometrical representation of complex numbers, because $c_\nu \sin (\nu x + \alpha_\nu)$ is the imaginary part of $c_\nu \exp i\, (\nu x + \alpha_\nu)$; our vector in the harmonic dial represents the "complex amplitude" $c_\nu \exp i\alpha_\nu$, the factor $\exp i\nu x$ being common to all waves of frequency ν. This is the connection to the electrotechnical diagrams used for describing alternating currents (see, for instance, 17, part 1, of the Handbuch der Experimentalphysik, Leipzig, 1934). For geophysical purposes, however, the special form of diagram described as harmonic dial is clearer.

up for $\rho=1$ to r, the terms on the right hand yield successively (the formulae applied being cited in brackets in each row)

$\sum_{\rho=1}^{r} y_\rho^2 = r\zeta^2 + ra_0^2$ After (11.2)

$\sum_{\rho=1}^{r} \phi(x_\rho)^2 = ra_0^2 + (r/2)\sum_{\nu=1}^{k}(a_\nu^2+b_\nu^2)$ After (*A1*.7) to (*A1*.9)

$\sum_{\rho=1}^{r} \phi^*(x_\rho)^2 = r(a_0^*)^2 + (r/2)\sum_{\nu=1}^{k}[(a_\nu^*)^2+(b_\nu^*)^2]$ After (*A1*.7) to (*A1*.9)

$-2\,\Sigma\, y_\rho\phi(x_\rho) = -2\,ra_0^2 - r\sum_{\nu=1}^{k}(a_\nu^2+b_\nu^2)$ After (*A1*.3)

$-2\,\Sigma\, y_\rho\phi^*(x_\rho) = -2\,ra_0a_0^* - r\sum_{\nu=1}^{k}(a_\nu a_\nu^*+b_\nu b_\nu^*)$ After (*A1*.3)

$2\,\Sigma\, \phi(x_\rho)\phi^*(x_\rho) = 2\,ra_0a_0^* + r\sum_{\nu=1}^{k}(a_\nu a_\nu^*+b_\nu b_\nu^*)$ After (*A1*.7) to (*A1*.9)

Therefore, the average square residual is, if we use (*A1*.5)

$$(A1.12)\qquad \sum_{\rho=1}^{r}[y_\rho - (\phi(x_\rho)+\phi^*(x_\rho))]^2/r = \zeta^2 - \eta_k^2 + (a_0^*)^2 + \sum_{\nu=1}^{k}[(a_\nu^*)^2 + (b_\nu^*)^2]/2$$

The minimum value of the right-hand side is (*A1*.6), if $a_0^* = a_1^* = b_1^* = \ldots = a_k^* = b_k^* = 0$. The case of $a_{r/2}$, for r even, is adequately covered by the proof.

Incidentally, (*A1*.12) proves a *corollary* to our main theorem. Thus if we require to approximate $f(x)$ by a sum of sine- and cosine-functions of frequency $\nu<r/2$, in which some of the frequencies are omitted (for example, if $r=12$, and we require approximation by $a_0+a_3\cos 3x+b_3\sin 3x+b_5\sin 5x$), the formulae (*A1*.3) remain valid, and (*A1*.6) also, if only the coecffiients actually used are inserted (in our example, $s^2=\zeta^2-(a_3^2+b_3^2+b_5^2)/2)$. This may be proved by putting the coefficients of ϕ^* equal to the negative coefficients (*A1*.3) of the terms omitted, and applying (*A1*.12).

The proof given here does not make use of differential calculus, at the same time furnishing the corollary mentioned.

A2. Fourier series for continuous function, and harmonic coefficients for equidistant ordinates: *Theorem II*—A continuous function f(x) between $x=0$ and 2π may be developed into an infinite Fourier series

$$(A2.1)\qquad \mathrm{f}(x) = A_0 + \sum_{\nu=1}^{\infty}(A_\nu\cos\nu x + B_\nu\sin\nu x)$$

implying that f(x) complies with the conditions necessary for this development. Furthermore, the r equidistant ordinates $\mathrm{f}(x_\rho)=y_\rho$ may, by (*A1*.3), be represented by the finite series $\phi(x)$ (*A1*.1), with harmonic coefficients a_ν, b_ν $(\nu<r/2)$. Then

$$(A2.2)\qquad a_0 = A_0 + A_r + A_{2r} + A_{3r} + \ldots$$

$$(A2.3)\qquad \begin{cases} a_\nu = A_\nu + A_{r-\nu} + A_{r+\nu} + A_{2r-\nu} + A_{2r+\nu} + \ldots \\ b_\nu = B_\nu - B_{r-\nu} + B_{r+\nu} - B_{2r-\nu} + B_{2r+\nu} - \ldots \end{cases}$$

and, for r even,

$$(A2.4)\qquad a_{r/2} = A_{r/2} + A_{3r/2} + A_{5r/2} + \ldots$$

Proof—In order to avoid excessive use of indices, the general proof may be abstracted from the following example: Put $r=12$, $\nu=5$: then $x_1=30°$, $x_\rho=\rho 30°$, $\cos\ (r-\nu)x_\rho=\cos\ (12-5)\rho 30°=\cos\ (\rho 360°-\rho 150°)$ $=\cos\ \rho 150°=\cos\ 5x_\rho=\cos\ \nu x_\rho$. Similarly, $\sin\ (r-\nu)x_\rho=-\sin\ \nu x_\rho$, $\sin\ (r+\nu)x_\rho=\sin\ \nu x_\rho$. Therefore, the finite series $\phi(x)$, with coefficients given by ($A2.2$) to ($A2.4$), has, for $x=x_\rho$, ordinates equal to those of $f(x)$.

Take as an example $r=3$ and $\nu=1$. Then $a_1=A_1+A_2+A_4+A_5+\ldots$. In the analysis of annual values, a wave A_2 of frequency 2 in 3 years, that is, of period 1.5 years, can be mistaken for a wave A_1 of frequency 1 in 3 years, that is, of period 3 years. The reason is obvious since $\cos x$ and $\cos 2x$ have, for $x=0°$, $120°$, and $240°$, the same numerical values, namely, 1, -0.5, and -0.5.

A3. Smoothing—From a continuous function $f(x)$ with the period 2π, that is, $f(x)=f(x+2\pi)$, a smoothed function $g(x)$ may be derived by ascribing to each abscissa x the average of $f(x)$ for the interval $(x-\beta)$ to $(x+\beta)$, that is, $g(x)=\int_{-\beta}^{+\beta} f(x+\xi)d\xi/2\beta$. Then the Fourier series of $g(x)$ is

$$(A3.1)\qquad g(x)=A_0+\sum_{\nu=1}^{\infty}(A_\nu \cos \nu x+B_\nu \sin \nu x)(\sin \nu\beta/\nu\beta)$$

If we plot a harmonic dial (section 6) for the vectors of sine-waves of period $2\pi/\nu$, that is for frequency ν, this equation means that the vector for $g(x)$ has the same direction, or the same phase, as that for $f(x)$, but the amplitude in $g(x)$ is reduced in the ratio $\sin \nu\beta/\nu\beta$. This function has been plotted (with $\delta=\nu\beta$) in Figure 19A. Negative sign of $\sin \nu\beta/\nu\beta$ means here reversal of phase, for instance: Average of $f(x)=\sin x$ for $\beta=3\pi/2$, that is, when smoothed over intervals of length 3π, $g(x)=-(2/3\pi)\sin x$.

Proof—Integrate each term of $f(x)$ in ($A2.1$): for instance, the integral of $\cos \nu(x+\xi)$ over $\xi=-\beta$ to $+\beta$ is $(1/\nu)\ (\sin \nu\ (x+\beta)-\sin \nu\ (x-\beta))$ $=(1/\nu)\ 2 \sin \nu\beta \cos \nu x$, and division by 2β gives the average $(\sin \nu\beta/\nu\beta)$ $\cos \nu x$.

Application—Hourly means in terrestrial magnetism (day$=360°$, $\beta=7.5°$), monthly means (year$=360°$, $\beta=15°$), etc. In practice, for instance, the hourly means are submitted to harmonic analysis as if they were equidistant values observed at the half-hours, and then the harmonic amplitudes are corrected to "instantaneous values" by multiplication with $\nu\beta/\sin \nu\beta$. This procedure neglects the possibility of higher frequencies in $f(x)$ than $r/2$, discussed in section $A2$; this is, however, generally not serious because, if r is not too small, the waves with frequencies above $r/2$ are very much reduced by smoothing.

A4. Non-cyclic correction—The values of the ordinates for $x=0$

and 2π may be y_0 and y_r. For r ordinates, our form of harmonic analysis (section 5 and section *A1*) considers only the ordinates y_1 to y_r, giving of course, $\phi(0)=y_r$ instead of y_0. If

$$y_r-y_0=d \text{ (non-cyclic change)} \tag{A4.1}$$

we can apply a non-cyclic correction by adding, before harmonic analysis, an appropriate linear function, namely, adding to y_ρ $(\rho=0, 1, \ldots, r)$ the value

$$(d/2)-(d\rho/r) \tag{A4.2}$$

If we submit these corrections to harmonic analysis, entering them for y_ρ in (*A1*.3), we obtain harmonic coefficients which we may call $\triangle a_\nu$ and $\triangle b_\nu$. Actual calculation gives the value of $\triangle a_0=-d/2r$. In order to obtain $\triangle a_\nu$ and $\triangle b_\nu$, we can, because of (*A1*.7), omit the constant part $(d/2)$ and consider only $(-d\rho/r)$. Inserting this value in (*A1*.3), we obtain for $\triangle b_\nu$, putting $\epsilon=2\pi\nu/r$

$$\triangle b_\nu=(2/r)\sum_{\rho=1}^{r}(-d/r)\ \rho \sin \rho\epsilon$$

$$\begin{aligned}
\triangle b_\nu\ (-r^2/d)\ \sin(\epsilon/2) &= \sum_{\rho=1}^{r} 2\rho \sin \rho\epsilon \sin(\epsilon/2) = \\
&\sum_{\rho=1}^{r}\{\rho \cos(\rho-1/2)\epsilon-\rho\cos(\rho+1/2)\epsilon\} \\
= \sum_{\rho=1}^{r}\{(\rho-1/2)\cos(\rho-1/2)\epsilon &-(\rho+1/2)\cos(\rho+1/2)\epsilon+ \\
&[\cos(\rho-1/2)\epsilon]/2+[\cos(\rho+1/2)\epsilon]/2\} \\
= \sum_{\rho=1}^{r}\{(\rho-1/2)\cos(\rho-1/2)\epsilon &-(\rho+1/2)\cos(\rho+1/2)\epsilon\}+ \\
&\cos(\epsilon/2)\sum_{\rho=1}^{r}\cos\rho\epsilon
\end{aligned}$$

By this rearrangement (known as "partial summation"), we can find the sum. The last sum vanishes because of (*A1*.7), and if we write out the successive terms for $\rho=1, 2, \ldots, r$ in the first sum we see that the positive and negative parts cancel, and only two remain from the terms with $\rho=1$ and $\rho=r$, namely, $\cos(\epsilon/2)/2-(r+1/2)\cos(r+1/2)\epsilon$, or $-r\cos(\epsilon/2)$ [since $r\epsilon=2\pi\nu$, and therefore $\cos(r+1/2)\epsilon=\cos(\epsilon/2)$]. Therefore, $\triangle b_\nu=(d/r)\cot(\epsilon/2)$. $\triangle a_\nu$ can in the same way, by multiplying by $\sin(\epsilon/2)$, be found as $-d/r$.

The non-cyclic correction can therefore be applied in the following simple way: Compute a_0, a_ν, b_ν from the given ordinates y_1 to y_r according to (*A1*.3), find the non-cyclic change $d=y_r-y_0$, add to a_0, a_ν, b_ν the corrections

$$\triangle a_0=-d/2r, \quad \triangle a_\nu=-d/r, \quad \triangle b_\nu=+(d/r)\cot(\pi\nu/r) \tag{A4.3}$$

Then $(a_0+\triangle a_0)$ is the arithmetic mean $(y_0/2+y_1+y_2+\ldots+y_{r-1}+y_r/2)/r$, and $(a_\nu+\triangle a_\nu)$ and $(b_\nu+\triangle b_\nu)$ are the harmonic coefficients of corrected ordinates obtained by adding to the given ordinates a linear function which makes the ordinates for $x=0$ and $x=2\pi$ equal.

The general formulae ($A4.3$) give, for $r=24$ and $r=12$, the corrections computed numerically by C. C. Ennis[37] (whose $C=y_0-y_r=-d$).

If the number r of the ordinates becomes infinite, the formulae ($A4.3$) become $\Delta a_0=\Delta a_\nu=0$, and, because $x/\sin x$ becomes 1 for $x=\pi\nu/r$ decreasing to 0, $\Delta b_\nu=d/\pi\nu$. This is of course nothing but the coefficient of the Fourier series for the continuous linear function $(d/2\pi)$ $(\pi-x)$, into which the non-cyclic correction ($A4.2$) is transformed by $r=\infty$, namely

$$(A4.4) \qquad (d/2\pi)\ (\pi-x)=(d/\pi)\ \{\sin x/1+\sin 2x/2+\sin 3x/3+\ldots +\sin \nu x/\nu+\ldots\}$$

This function is, by the way, discontinuous at $x=0$ and 2π, changing suddenly by the amount d. Finite partial sums of ($A4.4$) up to frequency ν exhibit therefore, near the discontinuity, the systematic lack of approximation known as *Gibbs' phenomenon*.[38] This is of little importance in geophysical applications, except as a warning that abrupt changes in the given function $f(x)$ can only be represented by including sine-waves of high frequency in the approximating series $\phi(x)$.

A5. Harmonic analysis and correlation—The correlation-coefficients between the given ordinates y_ρ (or their deviations $z_\rho=(y_\rho-a_0)$ from their arithmetic mean a_0) and the ordinates of the cosine-wave $\cos \nu x_\rho$ or the sine-wave $\sin \nu x_\rho$ are, respectively, $a_\nu/(\zeta\sqrt{2})$ and $b_\nu/(\zeta\sqrt{2})$, where ζ is the standard deviation of the y_ρ or z_ρ. [Indeed, the numerator of the correlation-coefficient is $\Sigma z_\rho \cos \nu x_\rho$, or because of ($A1.7$) and ($A1.3$), $\Sigma y_\rho \cos \nu x_\rho=(r/2)a_\nu$, and the denominator is the square root of the product Σz_ρ^2 $(=r\zeta^2)$ times $\Sigma \cos^2 \nu x_\rho$ $(=r/2$, because of ($A1.8$)).] Harmonic analysis can therefore be conceived as computation of correlation-coefficients.

Another relation to correlation can be seen in the formulae used in deriving ($A1.12$), because they can be interpreted for the calculation of the correlation-coefficient of two sets of ordinates $\phi(x_\rho)$ and $\phi^*(x_\rho)$ from the respective harmonic coefficients.

A6. The method of exhaustion—From ($A3.1$), it follows that the smoothed function $g(x)$ does not contain any periods p for which $\sin \nu\beta=0$, that is, $\nu\beta$ is an entire multiple $m\pi$ of π, or, since the length of the period $p=2\pi/\nu$, no periods of lengths $p=2\beta/m$, for which the smoothing interval 2β is an entire multiple mp; adjacent periods are weakened. If, therefore, we form the difference $d(x)=f(x)-g(x)$, it contains the sine-waves of these periods in full intensity, and adjacent periods in nearly full intensity. That is, in $d(x)$, the periods with longer periods than 2β are suppressed in favor of the shorter periods, This process and several similar processes like differentiation or integration, have been recommended therefore in order to help finding periodicities. However, though they may be useful for reconnaissance work and illustrative purposes,[39] they do not lend themselves readily to the application of the statistical tests for persistence, etc.

$d(x)$ may again be smoothed for a longer interval β_1, and this successive smoothing and difference-formation—a process which could be

[39] See the papers and diagrams given by M. et Mme. H. Labrouste, Ann. Inst. Phys. du Globe, Paris, **7**, 190 ff. (1929); **9**, 99-101 (1931); **11**, 93-101 (1933); Soixante-sixième Congr. d. Sociétés Savantes, 468-471 (1933); C.-R. Assemblée Lisbonne 1933, Union Géod. Géophys. Internat., Ass. Mag. Electr. Terr. Bull. No. 9, 292-295 (1934).

called method of *exhaustion*—has often been used as a substitute for harmonic analysis, not only because of the apparent saving of computing-labor but also because it has been thought to be independent of sine-waves (section 40). The method leads, however, in a roundabout way, to practically the same results as harmonic analysis, only obscuring its statistical aspect. The criticism[58] directed by H. H. Turner against an analysis of the sunspot-numbers, made by H. Kimura using the exhaustion-method, applies to a number of other papers.

A7. Refined computation of a persistent wave—As soon as the length of the period of a persistent wave is definitely known, its amplitude and phase can be obtained with an accuracy determined by the effective expectancy (section 36) and the number N of periods contained in the interval T of observation. Since, in general, N cannot be enlarged at will, the only possibility of increasing the accuracy is to lower the effective expectancy, for instance, by selecting, out of the whole interval T, suitable partial intervals with relatively smaller unperiodic variations. This has been done successfully in the computation of the lunar semi-diurnal waves in atmospheric pressure[47, 50]; fortunately, the error-estimates in the former paper[50] are based on monthly averages of the diurnal variation and need therefore not be revised after the effect of quasi-persistence has been detected. Of course, the selection of "quiet intervals" opens pitfalls which must be recognized, for instance, the curvature-effect (section 16). Particularly erroneous would be an attempt to compute an average vector from the single vectors with smaller amplitudes alone, for instance, from those in Figure 2 falling within the probable-error circle, because that would certainly lead to a systematic underestimate of the amplitude. But it would probably be admissible to compute the lunar 12-hour wave in pressure only from those days which have a small 24-hour wave. S. Chapman has proposed a scheme for a systematic reduction of the expectancy,[59] aiming at a corresponding increase of the accuracy with which the average sine-wave is obtained.

A8. Persistent waves with periodically changing amplitudes—The following formula is easily proved by applying (*A1*.11)

$$(A8.1) \qquad (c+2k\cos\mu x)\sin\nu x = c\sin\nu x + k\sin(\nu+\mu)\,x + k\sin(\nu-\mu)x$$

A wave with periodically changing amplitude is therefore equivalent to the sum of three persistent waves of different frequencies. This formula is much used in tidal theory, and can easily be demonstrated in the harmonic dial for frequency ν (two vectors of amplitude k, revolving with frequency μ in opposite directions around the end-point of the amplitude c).

Example—Terrestrial-magnetic activity, u_1-measure, 6-monthly wave,[48] amplitude varying in 11-year sunspot-cycle. Time t, origin at the beginning of a sunspot-year, increasing by 2π during one year. Then the 6-monthly wave is expressed by

$$[6.5+2.6\cos(t/11)]\sin(2t+261^\circ) = 6.5\sin(2t+261^\circ)$$
$$+1.3\sin(2t+(t/11)+261^\circ)+1.3\sin(2t-(t/11)+261^\circ)$$

[58]London, Mon. Not. R. Astr. Soc., 73, 543-552 (1913).
[59]Zs. Geophysik, 6, 396-420 (1930).

The frequencies (per year) of these terms are 2, 23/11, and 21/11, with periods 6.00, 5.74, and 6.29 months, respectively. Ordinary periodogram-analysis of a series of many years would therefore yield, besides the main 6-monthly wave, two waves of about one-fifth of its amplitude, and periods of 5.74 and 6.29 months; but this result differs only in form, not in physical content, from the statement of a 6-monthly wave of constant phase but variable amplitude.

Other examples are given by the case of solar diurnal waves with seasonally changing amplitudes (for instance, atmospheric temperature); the frequency of the solar diurnal wave is 365 per year and the frequency of the change of amplitude is 1 per year, so that the additional terms in ($A8$.1) have the frequencies 366 and 364 per year. The former has the period of a sidereal day, and this purely formal result has often been mistaken as a proof for influences of stars, etc.

Summary

(*a*) Every discussion of the physical causes of periodicities in geophysical and cosmical phenomena must be preceded by statistical studies testing the significance and reliability of these periodicities. This statistical viewpoint in the application of harmonic analysis was introduced by A. Schuster. The present paper gives, on the basis of the theory of probability, a new aspect and an improvement of these methods, generally called periodogram-analysis, or investigation of hidden periodicities. The scope of these results is not restricted to sine-waves.

(*b*) Following an introductory review of literature, harmonic analysis is discussed as a mathematical representation of time-functions, using vector-representation of sine-waves in the harmonic dial and the folding process as a graphical illustration of harmonic analysis. The degree of approximation between the given function and the sum of sine-waves is determined by the standard deviation of the residuals. The ordinary periodogram is introduced, and Pollak's periodogram for the international magnetic character-figure C is discussed.

(*c*) The generalized harmonic dial is introduced in order to prepare the transition from sine-waves to periodicities of other form. The nature of the non-cyclic variation and the selection- or curvature-effect, which is often misinterpreted, are discussed.

(*d*) The statistical laws for the random walk are described and applied, in various forms, to the harmonic dial and the folding process, using the conception of the summation-dial, expectancy c, and probability of chance (κ test, $1/\sqrt{n}$ law). For random fluctuations, the expectancy does not depend on the length of the period (equipartition of the variance).

(*e*) For geophysical phenomena, the expectancy depends definitely on the length of the period. This fact, often overlooked, is of decisive influence on tests for the reality of persistent periodicities, as demonstrated in several cases.

(*f*) Quasi-persistence, exhibited in limited sequences of successive waves, is described as a common phenomenon in geophysics and is measured by an index σ, the equivalent length of sequences. It affects the κ-test for persistent waves in so far as not the ordinary expectancy c but the effective expectancy $c\sqrt{\sigma}$ must be used. Because of interference,

adjacent periods are infected by quasi-persistence. Intermittent quasi-persistence indicates sub-periods of longer periods. In comparison with random conditions, and with respect to tests for persistence as well as the uncertainty of average sine-waves derived from a large material, quasi-persistence acts like a reduction of the number N of available observations to the effective number N/σ.

(*g*) The methods for testing geophysical phenomena with respect to periodicities are generalized for waves of other form than sine-waves. Typical examples are given for the characteristic, a diagram demonstrating, for an assumed length of period, in which way this period is contained in the observational material.

(*h*) In the appendix, the formulae for harmonic analysis of equidistant ordinates are derived, including the effects of smoothing and a new general formula for non-cyclic correction. The relations to correlation, the methods of exhaustion, and persistent waves with periodically changing amplitudes are discussed.

DEPARTMENT OF TERRESTRIAL MAGNETISM,
CARNEGIE INSTITUTION OF WASHINGTON.
Washington, D. C.