Motionally Induced Electric and Magnetic Fields in the Sea

THOMAS B. SANFORD

Woods Hole Oceanographic Institution, Woods Hole, Massachusetts 02543

Motionally induced electric and magnetic fields are investigated in the sea, crust, and mantle for large-scale low-frequency oceanic flows. It is shown that three-dimensional flows generate large-scale horizontal electric currents not present in two-dimensional motions. The resulting induced magnetic fields penetrate into the mantle inducing there electric currents. The degree of mutual induction between the ocean and the mantle depends on the parameter δ_m/L, the ratio of the electromagnetic skin depth of the mantle to the horizontal scale of the flow. For $\delta_m/L \gg 1$ there is little mutual induction whereas for $\delta_m/L \ll 1$ there is strong coupling between the ocean and the mantle. It is shown that the variable $\bar{\mathbf{V}}^*$, the conductivity-weighted, vertically averaged velocity is important in the generation of both local and large-scale electric currents. Expressions are derived showing that $\bar{\mathbf{V}}$, the vertically averaged velocity, can be determined from measurements of the induced electric and magnetic fields at the sea floor. Several special cases are calculated illustrating the influences of the mantle, the conducting sediments, and the horizontal scales of the motion.

Electric and magnetic fields are generated within ocean currents moving through the earth's magnetic field. The motionally induced electric field has been used by oceanographers to provide detailed information about the velocity structure of an ocean current. Quite often the electric field reveals structure that is difficult, if not impossible, to obtain by other methods. The method of towed electrode-pairs or geomagnetic electro kinetograph (GEK) [*von Arx,* 1950] and the use of submarine cables are the two principal examples of the electromagnetic approach to measuring oceanic flows. The induced magnetic field has received little attention. Recently, however, motionally induced electric and magnetic fields have been used for magneto-telluric soundings [*Larsen,* 1968].

The difficulties in using weak motionally induced electric fields in the sea are the low levels of the induced voltages and the method of interpreting the measurements. Recent advances in low-noise electronics have made the induced signals measurable over small distances. The interpretation, on the other hand, fails to account for certain features of actual ocean currents. This paper seeks to reveal more clearly the detailed relationship between the velocity field and the resulting electric and magnetic fields.

The most comprehensive study of the theory of the induced electric field in the sea has been made by *Longuet-Higgins et al.* [1954]. For steady two-dimensional flow they have used physical arguments to investigate the qualitative nature of the induced electric field and have presented detailed calculations for particular velocity distributions. The steady two-dimensional theory is, however, generally inadequate when applied to actual flows having variations in time and space. A three-dimensional ocean current flowing in a region of nonuniform total depth induces large-scale horizontal electric currents that do not arise in two-dimensional flow; moreover, these electric currents in the horizontal plane generate magnetic fields that are governed by the ocean current and the electrical conductivity of the earth. When the induced magnetic fields are time-dependent, they contribute to the electric field. The process by which magnetic fields interact with the motionally induced electric currents (called self and mutual induction) is an important aspect of the present analysis.

The purpose of this paper is to extend the theory of *Longuet-Higgins et al.* [1954] to three-dimensional time-dependent currents in a laterally unbounded ocean of nonuniform depth. To include mutual induction between the ocean

Contribution 2583 from the Woods Hole Oceanographic Institution.

and the mantle, an idealized model of the earth's electric conductivity structure is used. We restrict attention to large-scale low-frequency motionally induced electromagnetic fields. Hence, we ignore the influence of ionospheric disturbances in the sea and near ocean-continent boundaries [vide, *Cox et al.*, 1964; *Filloux*, 1967] and electromagnetic fields associated with surface gravity waves [*Weaver*, 1965].

The analytic procedure of this study is first to examine the electromagnetic equations in order to determine the conditions under which the quasi-static approximation can be made. We then compute the quasi-static electric fields and currents in the sea and sea bed. From the quasi-static electric currents the magnetic variations are calculated giving the approximate coupling between the electric and magnetic fields. The analysis involves multiple perturbation and iterative solutions as well as power series expansions. This seemingly awkward analytical approach is necessary because of the large number of length and time scales and conducting layers inherent in the model. This analysis is consistent with the idealization of the model, demonstrating the basic physics involved and providing a theory of broad applicability to oceanic flows.

NOTATION

- $\mathbf{A}$ magnetic vector potential, webers per meter.
- $\mathbf{a}$ induced magnetic vector potential, webers per meter.
- $\mathbf{B}$ magnetic induction, webers per meter^2.
- $\mathbf{b}$ induced magnetic induction, webers per meter^2.
- c speed of light in vacuum, meters per second.
- $\mathbf{D}$ electric displacement, coulombs per meter^2.
- D conductivity weighted depth, meters.
- $\mathbf{E}$ electric field, volts per meter.
- $\mathbf{F}$ earth's magnetic field, webers per meter^2.
- F magnitude of earth's magnetic field, webers per meter^2.
- H mean ocean depth, meters.
- h variations in depth from mean, meters.
- H_s depth of sediment–crust interface, meters.
- H_m depth of crust mantle interface, meters.
- H_a equivalent depth of mantle, meters.
- i imaginary unit $= (-1)^{1/2}$.
- $\mathbf{i}, \mathbf{j}, \mathbf{k}$ unit vectors in positive x, y, z directions.
- $\mathbf{J}$ electric current density, amperes per meter^2.
- $\mathbf{J}^*$ large-scale horizontal electric current density, amperes per meter^2.
- L typical horizontal scale, meters.
- $\mathbf{P}$ electric polarization, coulombs per meter^2.
- Q induction parameter.
- R range (3-dimensional), meters.
- r range (horizontal), meters.
- $\mathbf{V}$ velocity (V_x, V_y, V_z), meters per second.
- V_0 typical water speed, meters per second.
- x, y, z coordinates, meters.
- α, β wave numbers in x, y, meters^{-1}.
- γ horizontal wave number $= (\alpha^2 + \beta^2)^{1/2}$, meters^{-1}
- ∇ differential operator $= (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$, meters^{-1}.
- ∇_H horizontal differential operator $= (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j}$, meters^{-1}.
- δ electromagnetic skin depth, meters.
- ϵ, ϵ_0 electric permittivity in medium, in vacuum, farads per meter; $\epsilon_0 = (\mu_0 c^2)^{-1}$.
- ζ free surface elevation, meters.
- κ dielectric constant $= \epsilon/\epsilon_0$.
- λ conductance ratio of sediments and ocean.
- μ magnetic permeability, everywhere equal to $\mu_0 = 4\pi \times 10^{-7}$, henries per meter.
- ρ true charge density, coulombs per meter^3.
- τ decay time of magnetic field, seconds.
- ϕ electrostatic potential, volts.
- ω frequency, seconds^{-1}.

THE QUASI-STATIC ELECTRIC FIELD IN THE SEA

In this section the general electromagnetic equations and boundary conditions are specialized for an idealized model of the ocean and earth. A scaling of the electromagnetic equations yields the result that the induced electric field is quasi-static, being derivable from a scalar potential.

Maxwell's equations in rationalized mks units [*Panofsky and Phillips*, 1955] for a moving medium are

$$\nabla \cdot \mathbf{D} = \rho \qquad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3)$$

$$\nabla \times \mathbf{B} = \mu\left[\mathbf{J} + \rho\mathbf{V} + \frac{\partial \mathbf{D}}{\partial t} + \nabla \times \mathbf{P} \times \mathbf{V}\right] \quad (4)$$

with the constitutive equations

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \quad (5)$$

$$\mathbf{P} = \epsilon_0(\kappa - 1)(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \quad (6)$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \quad (7)$$

In accordance with (2) and (3) we let

$$\mathbf{B} = \nabla \times \mathbf{A}$$

with the restriction that

$$\nabla \cdot \mathbf{A} = 0$$

Then

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

The formal procedure would be to solve for ϕ and $\mathbf{A}$ under appropriate boundary conditions. A solution could be found only for explicit analytical forms of the velocity $\mathbf{V}(x, y, z, t)$ and the electrical conductivity $\sigma(x, y, z)$. However, the purpose of this paper is to determine the electric and magnetic fields associated with a general ocean current, assuming a conductivity model that includes only the important features of the actual structure. We seek not so much to solve exactly for $\mathbf{E}$ and $\mathbf{A}$ as to understand the influences of several important variables, such as bottom topography, variations in conductivity, and spatial and temporal variations in the velocity field.

Scaling. Several of the governing equations may be simplified. Let $\mathbf{B} = \mathbf{F} + \mathbf{b}$ where $\mathbf{F}$ is the steady geomagnetic field having zero divergence and curl, and $\mathbf{b}$ is the motionally induced magnetic induction. Also scale x, y by L, z by H, $\mathbf{V}$ by V_0, and $\mathbf{B}$ by F. It then follows that the magnitudes of $\mathbf{J}$ and $\mathbf{E}$ are $\sigma F V_0$ and $F V_0$ respectively, and for $H \ll L$, that the ratio of the four terms on the right side of (4) are as

$$1 : \frac{\epsilon_0 V_0}{\sigma H} : \frac{\omega\epsilon}{\sigma} : \frac{\epsilon_0 V_0}{\sigma H}$$

Upon substitution of the typical values ($\epsilon_0 = 10^{-11}$, $\kappa = 80$, $V_0 = 1$ m/sec, $H = 4 \times 10^3$ m, $\sigma = 4$ mho/m, $\omega \ll 1$ sec^{-1}) we see that the ratio of the first term, the conduction current $\mathbf{J}$ is to the other terms as at least $1 : 10^{-9}$. Thus we will replace (4) by

$$\nabla \times \mathbf{b} = \mu\mathbf{J} \quad (9)$$

which gives to the same order the equation for charge conservation

$$\nabla \cdot \mathbf{J} = 0 \quad (10)$$

The magnitude of $\mathbf{b}$ relative to $\mathbf{F}$ can be estimated from (9) to be

$$|\mathbf{b}/\mathbf{F}| \sim \mu\sigma H V_0 \sim 10^{-2}$$

We are now able to estimate the induced magnetic vector potential ($\mathbf{b} = \nabla \times \mathbf{a}$),

$$|\mathbf{a}| = \mu\sigma F V_0 H \mathcal{L}$$

where $\mathcal{L}$ is an as yet unspecified length scale except that it will be in the range $H < \mathcal{L} < L$. The precise value of $\mathcal{L}$ depends on the detailed structure of the electric currents and of the electrical conductivity of the crust and mantle. A further discussion of the point can be found in the later section dealing with the calculations of $\mathbf{b}$ and $\mathbf{a}$. For the present purpose of estimating $\partial\mathbf{a}/\partial t$, we take the largest value for $\mathcal{L}$. Then

$$\left|\frac{\partial \mathbf{a}}{\partial t}\right| \Big/ |\mathbf{E}| \sim \mu\sigma\omega H L \quad (11)$$

The magnetic induction will be small provided $\mu\sigma\omega H L < 1$. Since it is the variable ωL that is important let $\ell = 2\pi/L$. Substituting for μ, σ and H we find that $|\partial\mathbf{a}/\partial t|$ is small compared to $|\mathbf{E}|$ provided

$$\omega/\ell < 10 \text{ m/sec}$$

That is, we can initially neglect magnetic induction in broad ocean currents ($H \ll L$) provided there is no appreciable motion having a phase speed larger than 10 m/sec. This criterion will be refined later when we consider in more detail the electrical conductivity structure of the mantle.

Perturbation procedure. It is now possible to solve for ϕ and $\mathbf{a}$ by a perturbation analysis using the ordering parameter $\mu\sigma\omega H\mathcal{L}$. The governing equations for ϕ and $\mathbf{a}$ are

$$\nabla^2\phi = \nabla \cdot \mathbf{V} \times \mathbf{F}$$

$$\nabla \mathbf{a}^2 = -\mu \mathbf{J}$$

The procedure used is to solve for ϕ, taking $\mu\sigma\omega H\mathfrak{L} \ll 1$ (i.e., $\mathbf{a}^{(1)} = 0$); then, from (7),

$$\mathbf{J}^{(1)} = \sigma(-\nabla\phi + \mathbf{V}\times\mathbf{F})$$

with the corresponding magnetic vector potential given by

$$\mathbf{a}^{(2)} = \frac{\mu}{4\pi}\iiint_{-\infty}^{\infty} \frac{\mathbf{J}^{(1)}}{R}\, dx'\, dy'\, dz'$$

where

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$$

The next order correction to $\mathbf{J}$ is given by

$$\mathbf{J}^{(2)} = -\sigma\frac{\partial \mathbf{a}^{(2)}}{\partial t}$$

Higher order contributions to $\mathbf{J}$ and $\mathbf{a}$ are computed by this iterative scheme. The total fields are given by the sum of the terms

$$\mathbf{J} = \sum_{n=1}^{\infty} \mathbf{J}^{(n)} = \sigma\left(-\nabla\phi - \frac{\partial \mathbf{a}}{\partial t} + \mathbf{V}\times\mathbf{F}\right)$$

$$\mathbf{a} = \sum_{n=1}^{\infty} \mathbf{a}^{(n)}$$

The electromagnetic model used for the earth consists of an ocean of uniform conductivity σ_1 extending from the free surface to the sea bed, and a sea bed of uniform conductivity σ_2 extending to a plane boundary at $z = -H_s$. The crust between the sediments of the sea bed and the deep mantle boundary is assumed to have zero conductivity. The conductivity σ_m of the mantle is assumed to be that of highly conducting rocks. The permeability in each layer is taken to be that of free space.

Variations in the total water depth $H + \zeta - h$ have important consequences (Figure 1) and are included in the analysis. The mean depth is H, ζ is the departure of the sea surface from mean sea level ($z = 0$), and h is the departure of the sea floor from the mean depth ($z = -H$). In the deep sea $\zeta/H \ll 1$. We limit the model to $h/H \ll 1$.

Boundary conditions. The applied boundary conditions are such that the normal component of the electric-current density must vanish at the sea surface, the scalar potential and electric-current density are continuous across the ocean-floor interface, and the scalar potential vanishes far from the ocean current. For small depth variations, the outward normals, as shown in Figure 1, are

$$\mathbf{n}_1 \doteq \mathbf{k} - \nabla_H\zeta$$

$$\mathbf{n}_2 \doteq -\mathbf{k} + \nabla_H h$$

The boundary condition on the electric current at the surface is

$$\frac{\partial\phi_1}{\partial n_1} = \frac{\partial\phi_1}{\partial z} - \nabla_H\zeta\cdot\nabla_H\phi_1$$

$$= \mathbf{k}\cdot\mathbf{V}\times\mathbf{F} - \nabla_H\zeta\cdot\mathbf{V}\times\mathbf{F} \qquad (12)$$

Since ζ/L is small, the boundary condition may be applied at $z = 0$. Formally, we expand (12) in powers of ζ about $z = 0$, keeping only terms through ζ or its first derivatives. Then (12) becomes

$$\frac{\partial\phi_1}{\partial z} = \mathbf{k}\cdot\mathbf{V}\times\mathbf{F} + \nabla_H\cdot\zeta(\nabla_H\phi_1 - \mathbf{V}\times\mathbf{F})$$

$$- \zeta(\nabla^2\phi_1 - \nabla\cdot\mathbf{V}\times\mathbf{F}) \qquad (13)$$

On the right-hand side of (13) the second term is $-\nabla_H\cdot\zeta\mathbf{J}_1^{(1)}/\sigma_1$, while the third term is zero by (10). A similar expansion is made for the boundary condition at $z = -H + h$.

Separation of topographic effects. Due to the expansion of the boundary conditions the problem naturally divides into the following two parts: a first order problem involving the motionally induced field in an ocean of uniform depth and a second order problem involving the influences of total depth variations. Thus

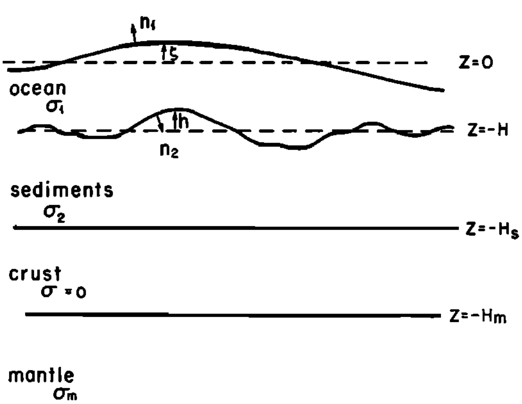

Fig. 1. Model of electrical conductivity of ocean and solid earth.

we look for solutions of the form $\phi = \phi' + \phi''$, where the primes refer to the first and second order solutions respectively. The ordering procedure follows directly from a scaling of (13) with the result that

$$\phi'' \sim \frac{h - \zeta}{L} \phi'$$

It is important to emphasize the distinction between the two ordering procedures involved here. The first based on the parameter $\mu\sigma\omega H\mathfrak{L}$ permits perturbation solutions to be found for the effects of magnetic induction on **E** and **b**. The second, based on the parameter $h - \zeta/L$, permits perturbation solutions to be found for the effects of topography. For example, to first order in $\mu\sigma\omega H\mathfrak{L}$, $\mathbf{E}^{(1)}$ is quasi-static (i.e., $\mathbf{E}^{(1)} = -\nabla\phi$). ϕ is then determined by another perturbation procedure in the parameter $h - \zeta/L$. This additional perturbation scheme is used only to solve for ϕ and not applied in the solutions for higher order terms in **E** and **b**.

Formal Solutions for Quasi-static Electric Field

The first order problem is that in the sea

$$\nabla^2\phi_1' = \nabla\cdot\mathbf{V}\times\mathbf{F} \tag{14}$$

and in the sea bottom

$$\nabla^2\phi_2' = 0 \tag{15}$$

with the boundary conditions

$$\frac{\partial\phi_1'}{\partial z} = F_y V_z - F_z V_y \qquad z = 0$$

$$\left.\begin{aligned} \frac{\partial\phi_1'}{\partial z} - \frac{\sigma_2}{\sigma_1}\frac{\partial\phi_2'}{\partial z} &= F_y V_z - F_z V_y \\ \phi_1' &= \phi_2' \end{aligned}\right\} z = -H \tag{16}$$

$$\frac{\partial\phi_2'}{\partial z} = 0 \qquad z = -H_s$$

Since the boundary conditions are on plane surfaces, we use double Fourier transforms over x and y. Any dependent variable, χ, will be transformed as

$$\langle\chi(\alpha, \beta, z, t)\rangle = \iint_{-\infty}^{\infty} \chi(x, y, z, t)e^{i(\alpha x+\beta y)}\, dx\, dy$$

with the inverse transform

$$\chi(x, y, z, t) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \langle\chi(\alpha, \beta, z, t)\rangle e^{-i(\alpha x+\beta y)}\, d\alpha\, d\beta$$

Through the use of the transforms, (14) becomes an ordinary differential equation having a general solution readily obtained by the method of variation of parameters. After satisfying the boundary conditions, the solutions to (14) and (15) are

$$\begin{aligned}\phi_1'(x, y, z, t) &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\left\{C_1\int_{-H}^{0}\left[\langle\mathbf{k}\cdot\mathbf{V}\times\mathbf{F}\rangle \sinh\gamma\xi - \langle\nabla_H\cdot\mathbf{V}\times\mathbf{F}\rangle\frac{\cosh\gamma\xi}{\gamma}\right]d\xi \right.\\ &\quad + \int_{-H}^{z}\left[\langle\mathbf{k}\cdot\mathbf{V}\times\mathbf{F}\rangle\cosh\gamma(z-\xi) \right.\\ &\quad \left.\left. + \langle\nabla_H\cdot\mathbf{V}\times\mathbf{F}\rangle\frac{\sinh\gamma(z-\xi)}{\gamma}\right]d\xi\right\} \cdot e^{-i(\alpha x+\beta y)}\, d\alpha\, d\beta\end{aligned} \tag{17}$$

$$\begin{aligned}\phi_2'(x, y, z, t) &= \frac{1}{4\pi^2}\iint_{-\infty}^{\infty}\left\{C_2\int_{-H}^{0}\left[\langle\mathbf{k}\cdot\mathbf{V}\times\mathbf{F}\rangle\sinh\gamma\xi \right.\right.\\ &\quad \left.\left. - \langle\nabla_H\cdot\mathbf{V}\times\mathbf{F}\rangle\frac{\cosh\gamma\xi}{\gamma}\right]d\xi\right\} \cdot e^{-i(\alpha x+\beta y)}\, d\alpha\, d\beta\end{aligned} \tag{18}$$

with

$$C_1 = \frac{\cosh\gamma(z+H) + p\sinh\gamma(z+H)}{\sinh\gamma H + p\cosh\gamma H}$$

$$C_2 = \frac{\cosh\gamma(H_s + z)}{\cosh\gamma(H_s - H)(\sinh\gamma H + p\cosh\gamma H)}$$

$$\gamma^2 = \alpha^2 + \beta^2$$

$$p = \frac{\sigma_2}{\sigma_1}\tanh\gamma(H_s - H)$$

The complexity of the formal solution tends to mask the physically relevant parameters. These solutions can be greatly simplified since we have assumed that $H/L \ll 1$. Therefore, the principal contributions to the integrals will be from the low values of γ for which $\langle \mathbf{V} \rangle$ is significant; that is $\langle \mathbf{V} \rangle$ is important over the range of α, β such that $(\alpha^2 + \beta^2)^{1/2}H \ll 1$. In this case the hyperbolic functions can be expanded in a power series with only the first one or two terms being retained. Thus the first term of importance in $\partial\phi_1'/\partial x$ is approximated as

$$\frac{-i\alpha \cosh \gamma(z + H)}{\sinh \gamma H + p \cosh \gamma H} \cdot \int_{-H}^{0} i(\alpha(\langle F_z V_y \rangle - \langle F_y V_z \rangle) + \beta(\langle F_x V_z \rangle - \langle F_z V_x \rangle)) \frac{\cosh \gamma\xi}{\gamma} \, d\xi$$
$$\doteq \frac{\alpha(\alpha(\langle F_z \bar{V}_y \rangle - \langle F_y \bar{V}_z \rangle) + \beta(\langle F_x \bar{V}_z \rangle - \langle F_z \bar{V}_x \rangle))}{\gamma^2(1 + \lambda)} \tag{19}$$

where

$$\lambda = \frac{\sigma_2(H_s - H)}{\sigma_1 H}, \qquad \bar{\mathbf{V}} = \frac{1}{H}\int_{-H}^{0} \mathbf{V} \, d\xi$$

The principal effect of the conducting sea bed is to reduce the terms involving $\bar{\mathbf{V}}$ by the factor $(1 + \lambda)^{-1}$ where the quantity λ is the ratio of the conductance of the sea bed to that of the sea water. We denote $\bar{\mathbf{V}}/(1 + \lambda)$ by the quantity $\bar{\mathbf{V}}^*$.

The right-hand side of (19) is simplified by adding and subtracting

$$\frac{\beta^2}{\gamma^2}(\langle F_z \bar{V}_y^* \rangle - \langle F_y \bar{V}_z^* \rangle)$$

which yields

$$\langle F_z \bar{V}_y^* \rangle - \langle F_y \bar{V}_z^* \rangle - \frac{\beta}{\gamma^2}(\alpha(\langle F_z \bar{V}_x^* \rangle - \langle F_x \bar{V}_z^* \rangle) + \beta(\langle F_z \bar{V}_y^* \rangle - \langle F_y \bar{V}_z^* \rangle))$$

The term multiplied by β/γ^2 is the Fourier transform of $i\nabla \cdot (F_z \bar{\mathbf{V}}^* - \mathbf{F}\bar{V}_z^*)$. Hence the approximation of $\nabla\phi_1'$ for small γ is (for convenience the superscript has been dropped from $\mathbf{J}$, i.e., $\mathbf{J}_1' = \mathbf{J}_1'^{(1)}$)

$$\nabla\phi_1' = \mathbf{V} \times \mathbf{F} - \mathbf{J}_1'/\sigma_1$$

where

$$\mathbf{J}_1'/\sigma_1 = \nabla \times \int_z^0 \mathbf{k} \times (\mathbf{V} - \bar{\mathbf{V}}^*) \times \mathbf{F} \, d\xi + \frac{\nabla \times \mathbf{k}}{2\pi} \iint_{-\infty}^{\infty} (\nabla \cdot \mathbf{k} \times \bar{\mathbf{V}}^* \times \mathbf{F}) \ln ((x - x')^2 + (y - y')^2)^{1/2} \, dx' \, dy'$$

and in the sea bed

$$\nabla\phi_2' = -\mathbf{J}_2'/\sigma_2 = -\nabla \times (H_s + z)\mathbf{k} \times \bar{\mathbf{V}}^* \times \mathbf{F} - \frac{\nabla \times \mathbf{k}}{2\pi} \iint_{-\infty}^{\infty} (\nabla \cdot \mathbf{k} \times \bar{\mathbf{V}}^* \times \mathbf{F}) \ln ((x - x')^2 + (y - y')^2)^{1/2} \, dx' \, dy'$$

In the second order topographic problem the potential is determined by the interaction of the first order solution with the topographic variations of the sea surface and sea floor. The equations to be solved are

$$\nabla^2\phi_1'' = 0$$

in the sea and

$$\nabla^2\phi_2'' = 0$$

in the sea bed. The boundary conditions are

$$\frac{\partial\phi_1''}{\partial z} = -\nabla_H \cdot \zeta \mathbf{J}_1'/\sigma_1 \qquad z = 0$$

$$\left.\begin{aligned} \frac{\partial\phi_1''}{\partial z} - \frac{\sigma_2}{\sigma_1}\frac{\partial\phi_2''}{\partial z} &= -\frac{1}{\sigma_1}\nabla_H \cdot h(\mathbf{J}_1' - \mathbf{J}_2') \\ \phi_1'' &= \phi_2'' \end{aligned}\right\} \qquad z = -H$$

$$\frac{\partial\phi_2''}{\partial z} = 0 \qquad z = -H_s$$

The solution follows in the same manner as for the first order problem

$$\phi_1''(x, y, z, t) = -\frac{1}{4\pi^2\sigma_1}\nabla_H \cdot \iint_{-\infty}^{\infty} \frac{(\cosh \gamma(z + H) + p \sinh \gamma(z + H))\langle \zeta \mathbf{J}_1' \rangle - \cosh \gamma z \langle h(\mathbf{J}_1' - \mathbf{J}_2') \rangle}{\gamma(\sinh \gamma H + p \cosh \gamma H)} \cdot e^{-i(\alpha x + \beta y)} \, d\alpha \, d\beta$$

$\phi_2''(x, y, z, t)$

$$= \frac{1}{4\pi^2\sigma_1} \nabla_H \cdot \iint_{-\infty}^{\infty} \frac{\cosh \gamma(H_s + z)(\langle h(\mathbf{J}_1' - \mathbf{J}_2')\rangle \cosh \gamma H - \langle \zeta \mathbf{J}_1' \rangle)}{\gamma \cosh \gamma(H_s - H)(\sinh \gamma H + p \cosh \gamma H)} \cdot e^{-i(\alpha x + \beta y)} \, d\alpha \, d\beta$$

Approximate Solutions for the Quasi-static Electric Field

The most useful expression for the quasi-static electric field is obtained after ϕ_1' and ϕ_1'' are combined and approximated for small wave numbers. Essentially, the second order solution contributes the terms needed to include ζ and h in the definition of $\bar{\mathbf{V}}$. The total potential gradient is (the superscript for $\mathbf{J}_1$ is reintroduced)

$$\nabla\phi_1 = \nabla\phi_1' + \nabla\phi_1'' = \mathbf{V} \times \mathbf{F} - \mathbf{J}_1^{(1)}/\sigma_1$$

where

$$\mathbf{J}_1^{(1)}/\sigma_1 = \nabla \times \int_z^{\zeta} \mathbf{k} \times (\mathbf{V} - \bar{\mathbf{V}}^*) \times \mathbf{F} \, d\xi$$

$$+ \frac{H(1 + \lambda)}{2\pi D} \nabla$$

$$\times \mathbf{k} \iint_{-\infty}^{\infty} \nabla \cdot \mathbf{k} \times \bar{\mathbf{V}}^* \times \mathbf{F} \ln r \, dx' \, dy'$$

$$+ \mathbf{k}\nabla_H \left\{ \frac{(z + H)\zeta - \left(1 - \frac{\sigma_2}{\sigma_1}\right) zh}{H(1 + \lambda)} \right\}$$

$$\cdot \frac{\nabla \times \mathbf{k}}{2\pi} \iint_{-\infty}^{\infty} \nabla \cdot \mathbf{k} \times \bar{\mathbf{V}}^* \times \mathbf{F} \ln r \, dx' \, dy'$$

$$- \nabla_H \left\{ \frac{1}{H(1 + \lambda)} \int_{-H}^{0} (\mathbf{k} \cdot \mathbf{V} \times \mathbf{F})\xi \, d\xi \right.$$

$$\left. + \int_{-H}^{z} \mathbf{k} \cdot \mathbf{V} \times \mathbf{F} \, d\xi \right\}$$

$$+ \mathbf{k}\nabla_H^2 \left\{ \frac{z + H}{H(1 + \lambda)} \int_{-H}^{0} (\mathbf{k} \cdot \mathbf{V} \times \mathbf{F})\xi \, d\xi \right.$$

$$\left. - \int_{-H}^{z} (\mathbf{k} \cdot \mathbf{V} \times \mathbf{F})(\xi - z) \, d\xi \right\}$$

$$+ \text{other terms of order } H^2/L^2. \qquad (20)$$

In the sea floor

$$\nabla\phi_2 = \nabla\phi_2' + \nabla\phi_2'' = -\mathbf{J}_2^{(1)}/\sigma_2$$

where

$$\mathbf{J}_2^{(1)}/\sigma_2 = \nabla \times (H_s + z)\mathbf{k} \times \bar{\mathbf{V}}^* \times \mathbf{F}$$

$$+ \frac{H(1 + \lambda)}{2\pi D} \nabla$$

$$\times \mathbf{k} \iint_{-\infty}^{\infty} \nabla \cdot \mathbf{k} \times \bar{\mathbf{V}}^* \times \mathbf{F} \ln r \, dx' \, dy'$$

$$+ \text{higher order terms in } H/L \qquad (21)$$

and

$$r = ((x - x')^2 + (y - y')^2)^{1/2}$$

$$D = H + \zeta - h + \frac{\sigma_2}{\sigma_1}(H_s - H + h)$$

$$\lambda = \frac{\sigma_2(H_s - H)}{\sigma_1 H}$$

$$\bar{\mathbf{V}}^* = \frac{1}{D} \int_{-H+h}^{\zeta} \mathbf{V} \, d\xi$$

Although not all terms in $\mathbf{J}_1^{(1)}$ are of the same order of magnitude, the expression is complete in the sense that $\nabla \cdot \mathbf{J}_1^{(1)} = 0$ to this order of approximation. The dominant terms in $\mathbf{J}_1^{(1)}$ are the first two, which are about an order of magnitude larger than the others. The topographic influences were not analyzed for the last two terms which are small in magnitude.

Discussion of the Solution

Consistent with the restriction of the analysis of low-frequency large-scale flows is the assumption that the vertical velocity is small compared with the horizontal velocity. That V_z is small leads to a useful simplification when we note

$$\mathbf{k} \times \mathbf{V} \times \mathbf{F} = F_z(V_x\mathbf{i} + V_y\mathbf{j}) - (F_x\mathbf{i} + F_y\mathbf{j})V_z$$

From here on we will neglect the term above involving V_z except when it is differentiated with respect to z. Thus the vector $\mathbf{V}$ is hereafter equal to $V_x\mathbf{i} + V_y\mathbf{j}$.

The first term in $\mathbf{J}_1^{(1)}$ represents local solenoidal electric currents produced by horizontal and vertical shear in the velocity field. The components of the corresponding potential gradient are

$$\frac{\partial\phi_1}{\partial x} = F_z\bar{V}_y^*$$

$$\frac{\partial \phi_1}{\partial y} = -F_z \bar{V}_x^*$$

$$\frac{\partial \phi_1}{\partial z} = F_y V_x - F_x V_y - \frac{\partial}{\partial x}\int_z^{\zeta} F_z(V_y - \bar{V}_y^*)\, d\xi + \frac{\partial}{\partial y}\int_z^{\zeta} F_z(V_x - \bar{V}_x^*)\, d\xi \quad (23)$$

The horizontal potential gradient divided by F_z is equal in magnitude and directed 90° to the right of the averaged velocity ($\bar{\mathbf{V}}^*$). Although this result, first derived by *Longuet-Higgins* [1949], has important oceanographic applications it has not been generally applied in the interpretation of GEK and submarine cable data. For example, the induced voltage across a submarine cable is usually interpreted by the relation (neglecting bottom conductivity)

$$\Delta\phi = \frac{F_z T_n}{H}$$

where T_n is the total transport (m³/sec) normal to the cable and H is the average depth of water. Although *Malkus and Stern* [1952] derived this relation for a uniform depth of ocean or channel, it is often applied to variable depth situations. A preferable expression for the induced potential is

$$\Delta\phi = F_z \int \bar{V}_n^*\, d\ell$$

A comparison of the two expressions for $\Delta\phi$ shows that H is not an actual depth but rather a velocity-weighted depth dependent on the distribution of the velocity over the actual depth. Hence H in this case must be considered time dependent. The large fluctuation in $\Delta\phi$ observed by *Wertheim* [1954] on the Key West–Havana submarine cable was probably caused by lateral shifts in the axis of the Florida current over variable depth rather than changes in transport as was assumed at the time.

Motionally induced electric fields have been measured at sea by a technique developed by von Arx called the geomagnetic electro kinetograph (GEK) [*von Arx*, 1950]. GEK observations consist of electrical measurements made between a pair of electrodes towed behind a vessel. Since the line connecting the electrodes is colinear with the velocity vector of the ship with respect to the surrounding water, this velocity component of the ship induces no measurable voltage. However, the surface current velocity can advect the electrode line through the vertical magnetic field and as a result a voltage is induced. Thus the GEK and similar techniques measure an apparent potential gradient $\nabla\phi_a$ given by

$$\frac{\partial \phi_a}{\partial x} = -F_z(V_y - \bar{V}_y^*)$$

$$\frac{\partial \phi_a}{\partial y} = F_z(V_x - \bar{V}_x^*)$$

where $\mathbf{V}$ is the local velocity at the sea surface or at the depth of measurement. It is general practice in the use of the GEK to replace the influence of $\bar{\mathbf{V}}^*$ through the use of the 'k factor' defined by

$$k = \frac{|F_z\mathbf{V}|}{|\nabla\phi_a|} = \frac{|\mathbf{V}|}{|\mathbf{V} - \bar{\mathbf{V}}^*|} \quad \text{at} \quad z = 0$$

Since the 'k factor' is an empirically determined aid to the interpretation of GEK data, it is recommended that the use of the 'k factor' be discontinued. The GEK should be interpreted as the vector difference between the surface and the vertically averaged velocities.

Although the conventional GEK measurement is obtained at the sea surface, it is possible to measure $\nabla\phi_a$ at all depths by a free-fall instrument. If we assume that the instrument is advected with the local horizontal flow, it is possible to measure $\nabla\phi_a$ or $\mathbf{V}(z) - \bar{\mathbf{V}}^*$ for all z. In this way the vertical variation or shear of the ocean current can be determined. An instrument has been developed that has successfully measured $\mathbf{V} - \bar{\mathbf{V}}^*$ from the top to the bottom of the ocean [*Drever and Sanford*, 1970.] Moreover, $\bar{\mathbf{V}}^*$ can be measured if the instrument comes to rest briefly at the sea floor. If $\bar{\mathbf{V}}^*$ does not change significantly during the period of measurement, the value obtained at the bottom can then be used to compute absolute values of $\mathbf{V}$ throughout the water column.

The vertical potential gradient is determined by the magnetic east–west velocity. The vertical electric current caused by the first term in expression for $\mathbf{J}_1^{(1)}$ is of the order of the ratio of the vertical to horizontal scales and is therefore generally negligible in broad ocean currents.

The second term in $\mathbf{J}_1^{(1)}$ represents the large-scale interaction between horizontally distant velocity fields. Whenever $\nabla \cdot F_z \bar{\mathbf{V}}^*$ is nonzero over large areas, horizontal electric currents flow in response to the spatially variable velocity field. This electric current arises from the large-scale flow although it is modulated by the local value of the factor H/D, the ratio of the mean depth to the actual depth in a conductivity weighted sense. The role played by free-surface and bottom topography should be emphasized. We have so far only required that the horizontal scale of the velocity be large compared with the depth. We can further require that the velocity obey the continuity equation $\nabla \cdot \mathbf{V} = 0$, that $\mathbf{V} \cdot \mathbf{n}_2 = 0$ at the bottom, and that $\mathbf{V} \cdot \mathbf{n}_1 = \partial \zeta / \partial t$ at the sea surface. With these conditions

$$\nabla \cdot \bar{\mathbf{V}}^* = -\frac{\partial \zeta / \partial t}{D} + \frac{\mathbf{V}(\zeta) \cdot \nabla \zeta}{D} - \frac{\bar{\mathbf{V}}^*}{D} \cdot \nabla D$$

Thus, in general, $\nabla \cdot F_z \bar{\mathbf{V}}^*$ will be nonzero within a time-dependent flow in an ocean of nonuniform depth. The relative importance of the above terms depends on the character of the flow. In shallow-water waves the term $\partial \zeta / \partial t$ is dominant, while the term $\bar{\mathbf{V}}^* \cdot \nabla D$ is important for meanders of the Gulf Stream over variable bottom relief. These electric currents are of large-scale and are independent of depth, having a magnitude not necessarily small compared with the electric current of a local origin as represented by the first term in expression for $\mathbf{J}_1^{(1)}$.

The convolution integral will arise often later so it is useful to denote this term by $\mathbf{J}_1^*/\sigma_1$ where

$$\mathbf{J}_1^*/\sigma_1 = \frac{\nabla \times}{2\pi} \mathbf{k} \iint_{-\infty}^{\infty} \nabla \cdot \mathbf{F}_z \bar{\mathbf{V}}^* \cdot \ln ((x - x')^2 + (y - y')^2)^{1/2} \, dx' \, dy'$$

Although the analytic form for $\mathbf{J}_1^*$ is useful in providing insight into the physics of large-scale interactions, this form is valid only for a laterally unbounded ocean. In general, one should expect that for localized flows distant from boundaries $\mathbf{J}_1^*$ is appropriate. Where flows within channels or along coasts occur a current analogous to $\mathbf{J}_1^*$ will arise but it will have a different analytic form.

The third term in $\mathbf{J}_1^{(1)}$ represents vertical electric currents that arise because of reflections of horizontal electric currents, expressed in the second term, from surface and bottom slopes.

The last two terms in $\mathbf{J}_1^{(1)}$ represent respectively the horizontal and vertical electric current density attributable essentially to horizontal shear in the velocity field.

Motionally Induced Magnetic Fields in the Sea

The motionally induced electric currents in the sea produce magnetic fields. The details of the magnetic field depend not only on the velocity structure of the oceanic flow but also on the electrical conductivity distribution of the ocean, crust, and mantle. In the previous sections we derived the scalar electric potential, assuming magnetic induction was negligible. The purpose of this section is to compute the magnetic vector potential in order to determine what if any are the consequences of neglecting magnetic induction.

An electromagnetic wave of frequency ω will penetrate into the mantle of conductivity σ_m to a depth scale δ_m given by

$$\delta_m = \left(\frac{2}{\mu \omega \sigma_m}\right)^{1/2}$$

The quantity δ_m is simply the electromagnetic penetration or skin depth. If $\delta_m/L \gg 1$ the mantle is transparent to the applied field while for $\delta_m/L \ll 1$ electric currents are induced in the mantle that oppose the penetration of the applied magnetic field.

That δ_m/L is the important quantity in this problem can be shown by using (2), (3), (4), and (7) to obtain

$$\frac{\partial \mathbf{b}}{\partial t} = \frac{1}{\mu \sigma_m} \nabla^2 \mathbf{b}$$

An applied magnetic field of horizontal scale L will attempt to penetrate the mantle to a depth of the order of L. The mantle has a decay time given by

$$\tau \sim \mu \sigma_m L^2$$

If the period of the applied magnetic field is T, the response of the mantle depends on the ratio T/τ. Clearly for $T/\tau \gg 1$ the mantle does not respond to the applied field and is therefore transparent. However, for $T/\tau \ll 1$ the mantle opposes the penetration of the applied field and can be considered to be opaque. The length and time ratios are related as

$$T/\tau \sim \delta_m^2/L^2$$

For $\delta_m/L \gg 1$ the mantle is unimportant and not required in a computation of the magnetic field in the sea. On the other hand, for $\delta_m/L \ll 1$ and $H_m/L \ll 1$ Larsen [1968] showed that the finite conducting mantle starting at H_m can be modeled by a superconducting layer at a depth of H_a where

$$H_a = H_m + (\mu\omega\sigma_m)^{-1/2}$$

The introduction of the superconducting layer is useful in that b_z must vanish at H_a. To satisfy this boundary condition an image current system (the negative of $\mathbf{J}$ in the ocean) is established in the region $-2H_a + H \geq z \geq -2H_a$. Then by symmetry b_z is zero at $z = -H_a$.

The magnetic vector potential is found by integration over all currents

$$\mathbf{a} = \frac{\mu}{4\pi} \iiint_{-\infty}^{\infty} \frac{\mathbf{J}}{R} \, dx' \, dy' \, dz'$$

where

$$R = ((x - x')^2 + (y - y')^2 + (z - z')^2)^{1/2}$$

It is convenient to express the vector potential in terms of a Fourier transform

$$\mathbf{a} = \frac{\mu}{16\pi^3} \int_{-\infty}^{\infty} dz' \cdot \iint_{-\infty}^{\infty} \langle \mathbf{J} \rangle \left\langle \frac{1}{R} \right\rangle e^{-i(\alpha x + \beta y)} \, d\alpha \, d\beta \qquad (24)$$

From (20) and (21) the significant terms in $\mathbf{J}$ are

$$\langle \mathbf{J}_1^{(1)} \rangle = \sigma_1 \left\{ \langle F_z \, \Delta V_y \rangle \mathbf{i} - \langle F_z \, \Delta V_x \rangle \mathbf{j} + i \frac{\langle \nabla \cdot F_z \bar{\mathbf{V}}^* \rangle}{\gamma^2} (\beta \mathbf{i} - \alpha \mathbf{j}) \right\}$$

$$\langle \mathbf{J}_2^{(1)} \rangle = -\sigma_2 \left\{ \langle F_z \bar{V}_y^* \rangle \mathbf{i} - \langle F_z \bar{V}_x^* \rangle \mathbf{j} - i \frac{\langle \nabla \cdot F_z \bar{\mathbf{V}}^* \rangle}{\gamma^2} (\beta \mathbf{i} - \alpha \mathbf{j}) \right\}$$

where

$$\Delta V_x = V_x - \bar{V}_x^*$$

and

$$\Delta V_y = V_y - \bar{V}_y^*$$

Since $1/R$ is symmetrical in x and y, its Fourier transform can be expressed as a Hankel transform [*Erdelyi et al.*, 1954]

$$\left\langle \frac{1}{R} \right\rangle = \frac{2\pi}{\gamma} e^{-\gamma |z - z'|}$$

Although variations in ζ and h can significantly influence the generation of electric currents, these variations have little effect on the magnetic fields produced. We now integrate (24) over z' term by term, ignoring ζ and h.

The first and second terms, involving ΔV_x and ΔV_y, produce locally strong but not far-reaching fields because their integral from $-H_s$ to 0 is zero. Since we assume $\gamma H \ll 1$, expanding $e^{-\gamma|z-z'|}$ in a power series and retaining the first nonzero terms, yields a contribution to $\mathbf{a}$ because of the local electric currents

$$\mathbf{a} = \frac{\mu}{2} \mathbf{k} \times \int_{-H_s}^{0} \sigma F_z \, \Delta \mathbf{V} \, |z - \xi| \, d\xi \qquad (25)$$

The degree of self induction produced by $\mathbf{a}$ is $\propto \mu\sigma\omega H^2$ which shows that for these electric currents $\mathcal{L} \sim H$. Since we have assumed that $\mu\sigma\omega HL < 1$ and that $H/L \ll 1$ we can ignore higher order contributions to $\mathbf{a}$. The corresponding magnetic induction is zero everywhere except in the ocean and sediments where it is given by

$$\mathbf{b}_1 = \mu\sigma_1 \int_z^0 F_z \, \Delta \mathbf{V} \, d\xi$$
$$\mathbf{b}_2 = \mu\sigma_2 (z + H_s) F_z \bar{\mathbf{V}}^* \qquad (26)$$

Larsen [1968] showed that waves of zero net transport, such as baroclinic tidal waves, produce negligible magnetic variations on the sea floor. However, (26) shows that at mid-depths the magnetic variations can be much greater than at the sea surface or bottom. The magnetic field at the surface or below the sediments is small because there are no net electric currents below or above these points. In this regard, it is useful to remember that these electric currents are solenoidal and that the magnetic fields generated are analogous to those of long solenoids.

The third term, involving the divergence of $\bar{\mathbf{V}}^*$, is the important term in the sense that it represents the large-scale horizontal electric currents that produce the correspondingly large-scale magnetic fields that penetrate into the crust and mantle. The contribution from this term to $\mathbf{a}$ in the ocean ($0 \geq z \geq -H$) is

$$\mathbf{a}^{(2)} = \frac{i\mu}{8\pi^2} \iint_{-\infty}^{\infty} \left\{ \langle \nabla \cdot F_z \bar{\mathbf{V}}^* \rangle \sum_{q=1}^{10} G_q \right\} \cdot \frac{(\beta \mathbf{i} - \alpha \mathbf{j})}{\gamma^4} e^{-i(\alpha x + \beta y)} \, d\alpha \, d\beta \qquad (27)$$

where

$$G_q = (-1)^q \sigma_q e^{-\gamma \zeta_q}$$

with

$$\zeta_q = 0, -z, 0, H + z, H + z, H_s + z, 2H_a - H + z, 2H_a - H_s + z, 2H_a + z, 2H_a - H + z$$

$$\sigma_q = \begin{cases} \sigma_1 & \text{for} \quad q = 1, 2, 3, 4, 9, 10 \\ \sigma_2 & \text{for} \quad q = 5, 6, 7, 8, \end{cases}$$

The evaluation of the above equation depends critically on the value of δ_m/L. It is assumed that $\gamma\zeta_q \ll 1$ for all q.

For $\delta_m/L \gg 1$, the mantle may be ignored and we need consider only the first six terms of ΣG_q in (27). On expanding G_q in a power series and taking the Fourier transform of the result, we obtain for $0 > z > -H$

$$\mathbf{a}^{(2)} = \frac{\mu\sigma_1 \bar{D}}{4\pi} \cdot \iint_{-\infty}^{\infty} \frac{\nabla \cdot F_z \bar{\mathbf{V}}^*((y-y')\mathbf{i}-(x-x')\mathbf{j})}{r} \, dx' \, dy' - \frac{\mu\sigma_1}{8\pi} (\bar{D}(H_s + H + 2z) + 2z^2 - H_s H) \cdot \iint_{-\infty}^{\infty} \frac{\nabla \cdot F_z \bar{\mathbf{V}}^*((y-y')\mathbf{i}-(x-x')\mathbf{j})}{r^2} \, dx' \, dy'$$

$$\bar{D} = H(1 + \lambda)$$

If we assume a time dependence of the form $e^{-i\omega t}$, we obtain for the comparison of $\partial \mathbf{a}/\partial t$ with $\nabla\phi$

$$\left| \frac{\partial \mathbf{a}}{\partial t} \right| \bigg/ |\nabla \phi| \sim \frac{LH}{\delta^2}$$

This relation states that $\mathcal{L} \sim L$ and provided $HL \ll \delta^2$ the electric field is quasi-static with little induction. A reasonable value for the conductivity of the sea is 4 mho/m, and for that of the mantle 0.4 mho/m [*Filloux*, 1967]. For order of magnitude comparisons, this gives $\delta_m = \delta$. Since we have already assumed that $\delta_m \gg L$ and $H/L \ll 1$, the condition $\delta^2 \gg HL$ is clearly satisfied. Higher order contributions to **a** are negligible.

Gulf Stream meanders probably are examples of motions that induce quasi-static electric and magnetic fields.

The magnetic field in the sea $0 \geq z \geq -H$ is

$$b_x = \frac{\mu \bar{D}}{2} \left(1 + \frac{2z}{\bar{D}} \right) J_{1\ y}^*$$

$$b_y = -\frac{\mu \bar{D}}{2} \left(1 + \frac{2z}{\bar{D}} \right) J_{1\ x}^* \qquad (28)$$

$$b_z = -\frac{\mu\sigma_1 \bar{D}}{4\pi} \iint_{-\infty}^{\infty} \frac{\nabla \cdot F_z \bar{\mathbf{V}}^* \, dx' \, dy'}{r} + \frac{\mu\sigma_1}{4} \cdot (\bar{D}(H_s + H + 2z) + 2z^2 - H_s H) \nabla \cdot F_z \bar{\mathbf{V}}^*$$

For $\delta_m/L \ll 1$, we must include all terms of $\sum G_q$ and, as before, expand G_q in a power series and take the Fourier transform of the terms. In this case $(0 \geq z \geq -H)$

$$\mathbf{a}^{(2)} = \frac{\mu}{2} \{ (2H_a - H_s - H) \bar{D} - z^2 + H_s H \} \mathbf{J}_1^*$$

The amount of mutual induction between the ocean and the mantle depends on the quantity $\mu\sigma_1\omega H_a \bar{D}$. We have assumed that $\mu\sigma_m\omega L^2 \gg 1$ and that $\mu\sigma_1\omega H\mathcal{L} < 1$. Clearly $\mathcal{L} \sim H_a$ and the procedure used so far is valid only if $\mu\sigma_1\omega H_a \bar{D} < 1$. That is, the method of treating the problem as quasi-static initially requires that $\mu\sigma_1\omega H_a \bar{D}$ be less than unity.

By using this magnetic vector potential and the expression for $\nabla\phi$, Ohm's Law yields for the current density (assuming time dependence as $e^{-i\omega t}$)

$$\mathbf{J}_1^{(2)} = \frac{i\omega\mu\sigma_1}{2} \cdot \{ (2H_a - H_s - H) \bar{D} - z^2 + H_s H \} \mathbf{J}_1^*$$

with a similar, but negative in value, term for the image region.

The quantity $Q = \mu\omega\sigma_1 H_a \bar{D}$, the degree of mutual-induction, is plotted in Figure 2 versus frequency ω. The computations were made using *Filloux*'s [1967] estimates of $H_m = 30$ km and $\sigma_m = 0.4$ mho/m for the oceanic mantle with $H = 4$ km and $\sigma_1 = 4$ mho/m ($\lambda = 0$) for the ocean. For periods of greater than several hours $Q < 0.4$. So the preceding analysis is now understood to be appropriate not only for large-scale ($L \gg H_m$) but also for long-period flows. With these restrictions, the iteration may be repeated indefinitely giving

$$\mathbf{J}_1 = \sigma_1 \nabla \times \int_z^0 F_z(\mathbf{V} - \bar{\mathbf{V}}^*)\, d\xi + \frac{1+i(H\bar{D}+\lambda H_s \bar{D}+z^2)/\delta^2}{1-iQ} \mathbf{J}_1^* + \cdots \quad (29)$$

The resulting expressions for the magnetic vector potential above, within, and below the ocean are

$$\mathbf{a} = \frac{\mu[\bar{D}(2H_a - H_s - H) + H_s H]}{2(1 - iQ)} \mathbf{J}_1^* \qquad z \geq 0$$

$$\mathbf{a} = \frac{\mu[\bar{D}(2H_a - H_s - H) - z^2 + H_s H]}{2(1 - iQ)} \mathbf{J}_1^* \qquad 0 \geq z \geq -H \quad (30)$$

$$\mathbf{a} = \frac{\mu\left[2\bar{D}(H_a + z) - \frac{\sigma_2}{\sigma_1}(H_s + z)^2\right]}{2(1 - iQ)} \mathbf{J}_1^* \qquad -H \geq z \geq -Hs$$

To the above can be added the small but general term (25).

The magnetic induction that is derived from (30) is analogous to that in a long many-layered solenoid. In the latter case, the magnetic induction is zero outside the solenoid, increases linearly within the windings, and attains a constant value inside the core. The electric currents in the sea are broad having image currents in the mantle so locally the situation is much like that in the center section of a long solenoid.

THE USE OF ELECTRIC AND MAGNETIC FIELDS TO DETERMINE $\bar{\mathbf{V}}$

This section combines the results of the previous sections to show that $\bar{\mathbf{V}}$ can be determined from electromagnetic measurements. It has been shown that **E**, **J**, and **B** depend on the characteristics of the velocity field (both local and distant), frequency and the conductivities of the ocean, sediments, crust, and mantle. Because of the complex relationship between $\bar{\mathbf{V}}$ and the resulting fields, it is difficult to determine

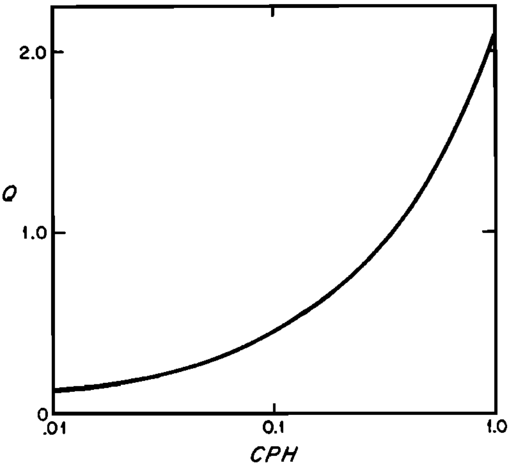

Fig. 2. Induction parameter versus frequency for conductivity model of earth (see text).

$\bar{\mathbf{V}}$ from **E**, **J** or **B** measurements used separately. However, simultaneous measurements of **E**, **J**, and **B** can be used to determine both $\mathbf{V}(z)$ and $\bar{\mathbf{V}}$.

The velocity profile can be obtained from a combination of $\mathbf{J}(z)$ and $\mathbf{E}(-H)$. The electric current density **J** can be measured by a free-fall electric field probe [*Drever and Sanford*, 1970] providing a relative velocity profile that can be made absolute by using the electric field at the sea floor. Since $\mathbf{k} \times \mathbf{E}(z) = \mathbf{k} \times \mathbf{E}(-H)$, then according to (7)

$$V_x \mathbf{i} + V_y \mathbf{j} = \frac{\mathbf{k} \times}{F_z}\left(\frac{\mathbf{J}(z)}{\sigma_1} - \mathbf{E}(-H)\right)$$

This scheme for measuring the absolute velocity profile is not dependent on the variable δ_m/L. Rather, as long as $H/L \ll 1$, the large-scale electric currents ($\mathbf{J}^*$) will be independent of depth regardless of self and mutual induction. This approach should be a very useful oceanographic technique because it is not dependent on precision navigation, is independent of the motion of the ship, and is capable of rapidly measuring velocity structure throughout the water column.

A useful technique for measuring $\bar{\mathbf{V}}$ for long periods is possible using simultaneous measurements of **E** and **B** at the sea floor.

Consider $\delta_m/L \gg 1$. Then from (26) and (28) E_x and b_y evaluated at $z = -H$ are

$$E_x = -F_z \bar{V}_y^* + J_x^*/\sigma_1$$

$$b_y = \mu\sigma_1 H\lambda F_z \bar{V}_y^* + \frac{\mu H}{2}(1-\lambda)J_x^*$$

Elimination of J_x^* yields

$$\bar{V}_y = \frac{1}{F_z}(-(1-\lambda)E_x + 2b_y/\mu\sigma_1 H)$$

and similarly

$$\bar{V}_x = \frac{1}{F_z}((1-\lambda)E_y + 2b_x/\mu\sigma_1 H)$$

The influence of the $\mathbf{J}^*$ can be eliminated but the determination of $\bar{\mathbf{V}}$ depends on an estimate of λ, the ratio of sediment to ocean conductances. The quantity λ can be estimated from geophysical data about the sediments. Alternatively, λ can be determined from the additional information from $\mathbf{J}(z)$. There are two measurements of $\bar{\mathbf{V}}$.

$$\bar{V}_y = \frac{1}{F_z}(J_x/\sigma_1 - E_x)$$

$$\bar{V}_y = \frac{1}{F_z}(-(1-\lambda)E_x + 2b_y/\mu\sigma_1 H)$$

Therefore

$$\lambda = \frac{J_x/\sigma_1 - 2b_y/\mu\sigma_1 H}{E_x}$$

For the case $\delta_m/L \ll 1$, (26), (29), and (30) evaluated at $z = -H$ yield

$$E_x = -F_z\bar{V}_y^* + \frac{1}{1-iQ}J_x^*/\sigma_1$$

$$b_y = \mu\sigma_1 H\lambda F_z\bar{V}_y^* + \frac{\mu H}{1-iQ}J_x^*$$

Elimination of J_x^* yields

$$\bar{V}_y = \frac{1}{F_z}(-E_x + b_y/\mu\sigma_1 H)$$

and similarly

$$\bar{V}_x = \frac{1}{F_z}(E_y + b_x/\mu\sigma_1 H)$$

Here the magnetic field provides the needed information to eliminate from the electric field the influences of the sediments (λ) and those of the mutual induction with the mantle (Q). It is not possible to solve for λ or Q because $\bar{\mathbf{J}}/\sigma_1$ provides no new information (e.g., $\mu H J_x = b_y$).

SEVERAL EXAMPLES OF THE INDUCED ELECTRIC AND MAGNETIC FIELDS WITH LARGE-SCALE LOW-FREQUENCY OCEANIC FLOWS

There is considerable interest in the study of large-scale low-frequency motions in the sea. Unfortunately, there are too few observations to define the motion adequately. The scarcity of reliable measurements is partly attributable to the difficulty in making long-term observations in the sea and to the absence of appropriate sensors. Because of the rich spectral content of the motion of the sea, it is generally difficult to measure large-scale low-frequency processes using sensors responding to the complexity of its local velocity field. However, it seems likely that electromagnetic measurements offer the possibility of reducing the influence of local effects and emphasizing the broad spatially averaged nature of ocean currents. This approach is well suited to studies, for example, of quasi-steady currents, such as the Gulf Stream, deep sea tides, and inertial and planetary waves.

The present analysis shows that the induced electric field can be a complicated function of the velocity field. The electric field responds to both the local and large-scale character of the flow. Presently it is easier to make electric, as opposed to magnetic, field measurements for long periods. However, it is important before electrical measurements are taken to determine the influence of the distant velocity structure relative to the locally generated electric field. A simple example should illustrate the electrical interaction of distant velocity fields. An ocean current of the form

$$V_x = 2V_0(1 + z/H)e^{-\ell|y|}e^{i(kx-\omega t)}$$

contains the relevant parameters ω, ℓ, k, V_0 and vertical shear, which determine the induced electric and magnetic response. The electric currents which vary with depth are controlled by the local vertical shear while those in the horizontal plane and independent of depth are produced by the divergence of the average velocity. The latter currents generated in regions where $kx - \omega t = m\pi$ interact with the adjacent regions $m - 1$ and $m + 1$. In this way current loops are produced with amplitude and shape governed by the values of k and ℓ.

Using this example of flow the only term in (20) that is not immediately determined is $\mathbf{J}^*$

where for $Q = 0$ and $\sigma_2 = 0$

$$\mathbf{J}_1{}^* = \frac{\sigma_1 F_z V_0}{2k} \nabla \times \mathbf{k}\psi^*$$

and

$$\psi^* = \frac{2k(\ell e^{-k|y|} - ke^{-\ell|y|})}{k^2 - \ell^2} e^{i[kx - \omega t + (\pi/2)]}$$

The electric and magnetic fields are then determined from (20) and (30).

The interaction between the two horizontal scales can be illustrated by the following three special cases ($Q = 0$ and $\sigma_2 = 0$):

1. A long-wavelength short-crested wave $k/\ell \ll 1$

$$E_x \sim \frac{k}{\ell} F_z V_0$$

$$E_y = F_z \bar{V}_x$$

2. A wave of equal scales, $k/\ell = 1$

$$E_x = \frac{F_z \bar{V}_x}{2} kye^{i\pi/2}$$

$$E_y = \frac{F_z \bar{V}_x}{2} (1 - k\,|y|)$$

3. A short-wavelength broad-crested wave, $k/\ell \gg 1$

$$E_x \sim \frac{\ell}{k} F_z V_0$$

$$E_y \sim \frac{\ell}{k} F_z V_0$$

For all cases

$$E_z = -F_y V_x$$

provided ℓ and k are small compared to H^{-1}.

The interpretation of these results is that in case 1 the flow is almost two-dimensional producing negligible $\mathbf{J}^*$ electric currents. On the other hand, case 2 demonstrates that when $k \sim \ell$, significant electric currents flow that establish an electric field in the direction of motion while reducing the transverse electric field. In case 3, the electric fields disappear because of large electric currents that tend to short-out the electric source.

The exchange of electric currents between distant portions of an ocean current may severely limit the usefulness of electrical measurements in large-scale barotropic flows. To further illustrate this point, consider the difference between the electric fields when $\mathbf{J}^*$ is important and when it is not; these situations correspond to cases 1 and 2. The normalized potential function

$$\tilde{\phi} = -\frac{\ell}{F_z V_0} \int_0^y E \cdot d\eta$$

is plotted in Figure 3 for case 1 ($k = 0$) and for case 2 ($k = \ell$) where $kx - \omega t = \pi$. Since ℓ is the same for both cases, the transport per unit width of the current is the same. The potential functions in Figure 3 show the influence of $\mathbf{J}^*$ electric currents. The conventional interpretation of measurements of the potential field is to

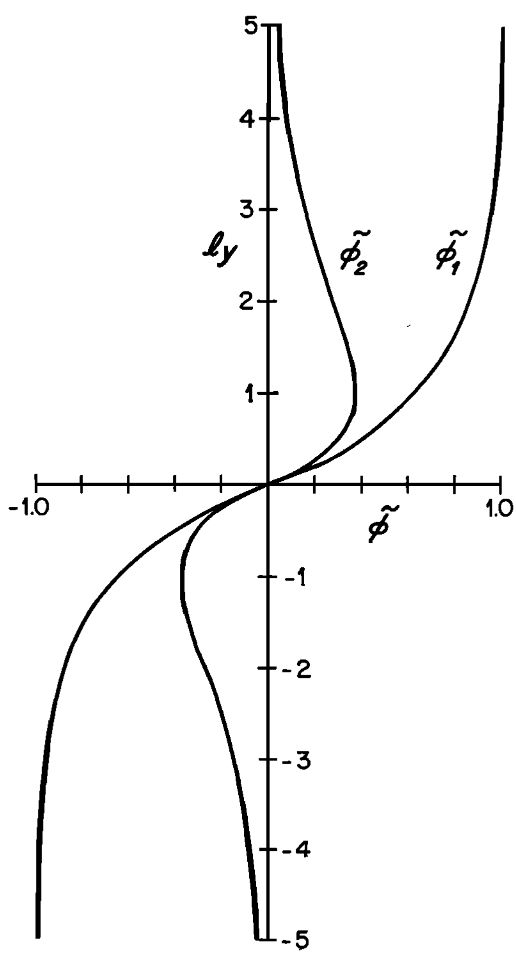

Fig. 3. Normalized electric potential functions for cases 1 and 2.

assume $k = 0$ (case 1), however, if $k \neq 0$ this procedure leads to considerable error. Another feature of case 2 is that because of the $\mathbf{J}^*$ electric current, the electric field will exist even where the local velocity is zero.

Provided $k \ll \ell$, electrical measurements provide useful information. As k approaches ℓ, the generated electric field is strongly dependent on the ratio k/ℓ. It is therefore expected that deep sea electrical measurements using submarine cables can be interpreted only in conjunction with other measurements or theoretical predictions regarding the horizontal structure of the flow.

For the purposes of the above examples it has been assumed that $Q = 0$ and $\lambda = 0$. Nonzero values of these parameters will have effects as indicated in (20) and (29). For $\delta_m \ll L$ or as in case 2 $k\delta_m \ll 1$, there will be electric currents in the ocean and mantle. Figure 4 shows the horizontal structure of $\mathbf{J}^*$ for case 2. Several streamlines are drawn for the function

$$\psi^* = (1 + k\,|y|)e^{-k|y|}e^{i[kx-\omega t+\pi/2]}$$

The current pattern is of large horizontal extent and is independent of depth. Allowing for the nonuniform vertical exaggeration, the similarity is evident between the currents and the magnetic induction in the ocean ('image ocean') system and in a finite length solenoid. The degree of mutual induction will determine the current density. The large-scale electric currents will be equal to $\mathbf{J}^*/1 - iQ$.

The magnetic induction in the direction of flow for case 2 is

$$b_x = \mu\sigma_1 F_z \int_z^0 (V_x - \bar{V}_x^*)\, d\xi - \frac{\mu\sigma_1 z F_z \bar{V}_x^*}{2(1 - iQ)}(1 + k\,|y|)$$

Let $\sigma_1 = 3$ mho/m, $\sigma_2 = 0$, $F_z = -0.4 \times 10^{-4}$ weber/m^2, $H = 5$ km, $Q = 0.4$, and $V_0 = 10^{-2}$ m/sec. The magnitudes of the first term, the second term, and the sum are presented in Figure 5 for $y = 0$ and $kx - \omega t = 0$. As expected, the electric currents caused by vertical shear produce magnetic fields wholly contained

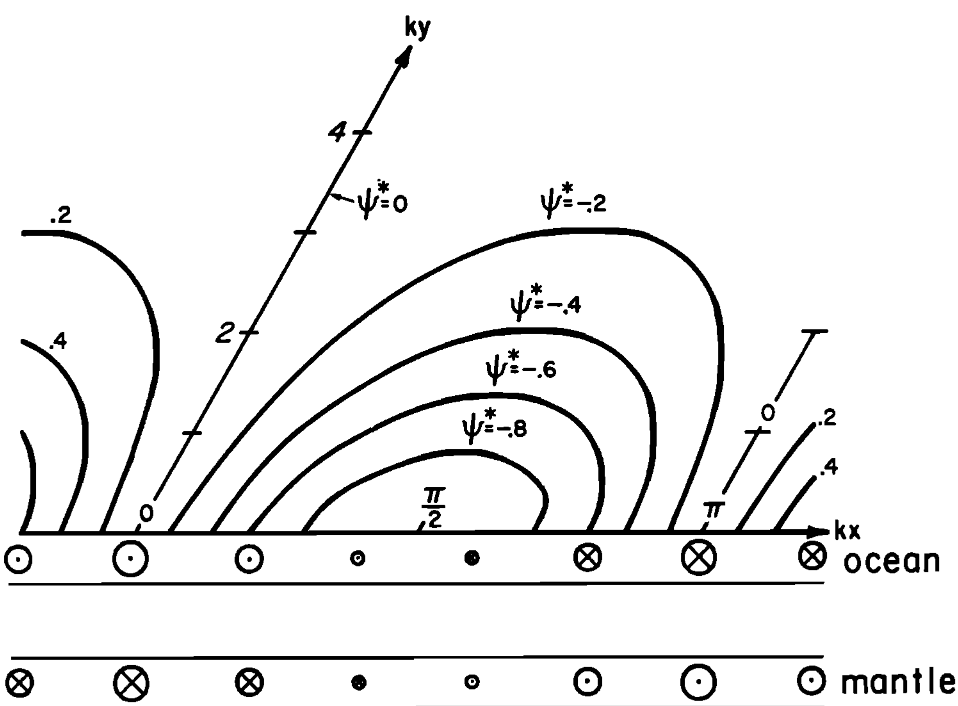

Fig. 4. Large-scale horizontal electric currents induced in the ocean and in the mantle for case 2. The quantity ψ^* is proportional to a stream-function for $\mathbf{J}^*$. The magnitude of $\mathbf{J}^*$ is shown by the diameter of the circles.

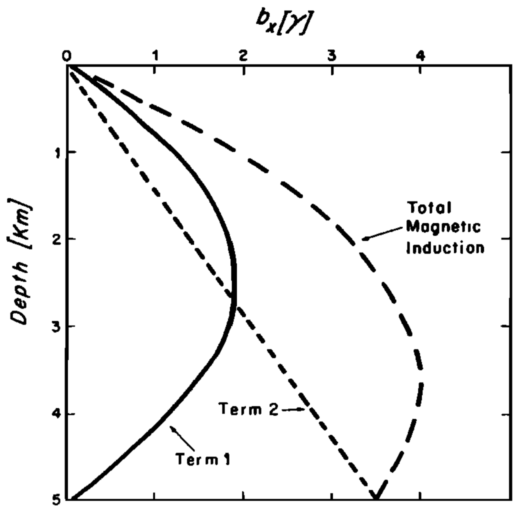

Fig. 5. Induced magnetic field component for case 2.

within the ocean while the electric currents caused by the average velocity generate fields that increase linearly with depth, remaining constant below the sea floor. For an average flow of 1 cm/sec the total magnetic field attains a value of about 4γ near the sea floor. On the other hand, the electric currents caused by vertical shear in the Gulf Stream generate horizontal magnetic components in excess of 100γ at a depth of several hundred meters.

Conclusions

This paper has examined the electromagnetic field that arises within ocean currents. An attempt has been made to include the important parameters. In an ocean current broad compared to the ocean depth, the variable $F_z\bar{\mathbf{V}}^*$ plays a dominant role. The principal electric currents and magnetic fields generated in the sea are classed as local fields dependent on the variable $F_z(\mathbf{V} - \bar{\mathbf{V}}^*)$ and large-scale fields governed by the horizontal distribution of $\nabla \cdot F_z\bar{\mathbf{V}}^*$.

The electrical conductivity model used for the earth is crude, incorporating the gross features of the actual structure. The magnetic fields and their resulting induction are to be considered as approximations useful in estimating the importance of a highly conductive upper mantle. The magnetic fields and the degree of magnetic induction (self and mutual induction) within the ocean and between the ocean and mantle depend on the various time and space scales of the flow and on the electrical conductivity of the mantle. For $\delta_m/L \ll 1$ the mantle is transparent to the motionally induced magnetic field; little self-induction and no mutual-induction results. For $\delta_m/L \ll 1$ the mantle supports electric currents leading to significant mutual induction.

The present results suggest that considerable new information about ocean currents can be obtained from electric and magnetic measurements. Measurements of the horizontal electric and magnetic fields on the sea floor can permit studies of large-scale barotropic flows. Also, the measurement of the electric field as sensed by a free-fall instrument can provide new information on the detailed vertical structure of ocean currents. These new observational techniques can provide data that is difficult to obtain by other means.

When considering measurements of the induced electromagnetic fields, it is important to remember the specific restrictions of the theory to low-frequency large-scale flows. In general a flow is composed of motions of many temporal and spatial scales. Certain components of a flow may not satisfy the restrictions. Moreover, magneto-telluric currents will occasionally produce electric and magnetic fields larger than the motionally induced fields. The influences of magneto-telluric currents can be reduced somewhat when magnetic and electric variations are measured. Induction within flows that do not satisfy the present theory, such as internal gravity waves, requires further analysis.

Appendix

The Influence of Sea Water Conductivity Variations on Electric Currents

The spatial variations of sea water conductivity will produce slight distortions in the potential field as represented by (20). A local flow will induce electric currents which circulate through adjacent regions of different velocity and conductivity. The effective resistance along any current path is determined by the velocity and conductivity along that path. For example, in the uniform conducting sea treated previously, the important quantity in the J equations was the vertically averaged velocity. The influence of conductivity was derived by *Longuet-Higgins et al.* [1954]

for two-dimensional flows over an insulating sea bed. We now extend the treatment to three dimensions and include a conducting sea bed.

In general, the electrical conductivity of the sea varies continuously in space. Although the expressions for the electrical current density were derived for uniform conductivity, we now permit σ to change and require the changes in **J** so produced to be consistent with a general constraint derived from the continuity equation for **J**. Integrating the continuity equation for **J** from $z = 0$ to $z = -H_s$ yields

$$\nabla_H \cdot \int_{-H_s}^{0} J\, dz = 0$$

From (20) and (21) the divergence of the electric current density in the ocean and in the sea bed are

$$\nabla_H \cdot \mathbf{J}_1^{(1)} = \frac{\partial}{\partial x} \sigma_1 F_z (V_y - \bar{V}_y^*) - \frac{\partial}{\partial y} \sigma_1 F_z (V_x - \bar{V}_x^*)$$

$$\nabla_H \cdot \mathbf{J}_2^{(1)} = -\frac{\partial}{\partial x} \sigma_2 F_z \bar{V}_y^* + \frac{\partial}{\partial y} \sigma_2 F_z \bar{V}_x^*$$

Hence

$$\frac{\partial}{\partial x}\left(\bar{V}_y^* \int_{-H_s}^{0} \sigma\, d\xi - \int_{-H}^{0} \sigma V_y\, d\xi\right) - \frac{\partial}{\partial y}\left(\bar{V}_x^* \int_{-H_s}^{0} \sigma\, d\xi - \int_{-H}^{0} \sigma V_x\, d\xi\right) = 0$$

For the two-layer case where σ_1 and σ_2 are constant, $\bar{\mathbf{V}}^* = \bar{\mathbf{V}}/(1 + \lambda)$ and each of the terms in parentheses vanish. By induction, the general expression for the conductivity-weighted average velocity is

$$\bar{\mathbf{V}}^* = \frac{\int_{-H+h}^{\zeta} \sigma \mathbf{V}\, d\xi}{\int_{-H_s}^{\zeta} \sigma\, d\xi}$$

Acknowledgments. I am indebted to Dr. William von Arx and Dr. George Needler for valuable comments during the course of the work and for reading the manuscript.

This research was supported by the Office of Naval Research contract N00014-66-C0241.

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(Received October 28, 1970;
revised February 8, 1971.)